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ABSTRACT

This is the third part of a three-part SMSG algebra text for high school students. Chapter titles include: Truth Sets of Open Sentences; Graphs of Open Sentences in Two Variables; Systems of Equations and Inequalities; Quadratic Polynomials; and Functions. (MK)

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MATHEMATICS FOR HIGH SCHOOL

FIRST COURSE IN ALGEBRA (Part 3)

(preliminary edition)

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FIRST COURSE IN ALGEBRA

(Part III)

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Chapter 10

Truth Sets of Open Sentences

10 - 1. Equivalent open sentences. Let us recall some open sentences whose truth sets we found in earlier chapters.

Example 1. $3 + x = 7$ (Section 2 - 8). By guessing, we found that 4 was a solution, that is, a member of the truth set. Then we showed that any number different from 4 could not be a solution of the equation. Together, what did these two statements prove about the truth set of the equation? Do $3 + x = 7$ and $x = 4$ have the same truth set?

Example 2. $3x + 7 = x + 15$ (Section 6 - 5). Here we used the addition and multiplication properties of equality to show that every solution of $3x + 7 = x + 15$ is a solution of $2x = 8$, and every solution of $2x = 8$ is a solution of $x = 4$. Since the solution of $x = 4$ obviously is 4, we knew that the only possible solution of $3x + 7 = x + 15$ is 4. Can we therefore say that 4 is a solution of the original equation? How can we determine the truth set of the equation? Can you explain why $3x + 7 = x + 15$ and $x = 4$ have the same truth set?

In working with Example 1 we used some guesswork; in Example 2, guessing was eliminated by finding a much simpler equation, $x = 4$, that has two advantages: its truth set is easy to see, and its truth set contains every solution of $3x + 7 = x + 15$. On the other hand, either by starting with 4 and working backwards, or by substituting 4 into $3x + 7 = x + 15$, we find that every solution of $x = 4$ is a solution of $3x + 7 = x + 15$. Thus the two equations have exactly the same truth set, and the truth set of $3x + 7 = x + 15$ consists of exactly the one number 4.

Is this reverse argument necessary? If we build from our original equation a new simple equation, can we always be sure that the new truth set is exactly the truth set of the original? In all our examples so far, the new truth set has been the same as the original. Let us try a new example.

Example 3. $\sqrt{x} + 2 = x$.

If x is any number for which the equation is true, then

$\sqrt{x} = x - 2$, (Why?)

$x = (x - 2)^2$; By squaring both sides.

$x = x^2 - 4x + 4$, (Why?)

$0 = x^2 - 5x + 4$, (Why?)

$0 = (x - 4)(x - 1)$. (Why?)

Thus every solution of $\sqrt{x} + 2 = x$ is a solution of $0 = (x - 4)(x - 1)$. The truth set of this last equation is $\{1, 4\}$; give reasons for this. May we automatically assume, then, that $\{1, 4\}$ is the truth set of the original equation? What is the truth set of the original equation? Apparently the last truth set is not the same as the truth set of the original equation.

Exercises 10 - 1a. In each part of each of the following nine problems, a pair of equations is given. Show that every solution of the first equation is a solution of the second equation, noting at each step what you have done (for example, added -4 to each side, or multiplied each side by 2, or squared both sides, etc.) . Then see if, by reversing your steps (adding 4 to each side, multiplying by $\frac{1}{2}$, etc.) , you can show that every solution of the second equation must be a solution of the first. If not, try to find a solution of the second which is not a solution of the first.

1. (a) $4 - 2x = 10$; $-2x = 6$.
   (b) $-2x = 6$; $x = -3$.
2. (a) $12x + 5 = 10 - 3x$; $15x = 5$.
   (b) $15x = 5$; $x = \frac{1}{3}$.
3. (a) $x^2 - 4 = 0$; $(x - 2)(x + 2) = 0$.
   (b) $(x - 2)(x + 2) = 0$; $x - 2 = 0$ or $x + 2 = 0$.
4. (a) $x = 3$; $x^2 = 3x$.
   (b) $x^2 = 3x$; $x(x - 3) = 0$.
5. $x - 1 = 0$; $x^2 - 1 = 0$.
6. $x^6 - 4x^5 + x^3 + 1 = 0$; $(x^6 - 4x^5 + x^3 + 1)(x - 1) = 0$.
7. $|x| = 1$; $x^2 = 1$.
8. $x^2 - 3 = 4 + x^2$; $1 = 8$.
9. $x + \frac{1}{x - 2} = 2 + \frac{1}{x - 2}$; $x = 2$

There seems to be a fundamental difference between Example 2 and Example 3. In Example 2 the original equation and the final equation, $x = 4$, have exactly the same truth set; in Example 3 the original and the final equations do not have the same truth set. Thus, when we change equations, sometimes the truth set is changed. But suppose we knew ways of changing sentences so as always to leave the truth sets unchanged. Then we could use the following short and correct procedure for finding the truth set of a sentence. Change the sentence to simpler sentences, at each step being sure that the method for that step has left the truth set unchanged. Keep this up until we get a sentence whose truth set is obvious. This truth set is then the truth set of the original sentence.

Thus we want to concentrate on how to change a sentence to a new sentence in such a way that the truth set remains the same. We give a name to sentences related this way.

> Two sentences with the same truth set
> are said to be equivalent.

The question now is: What operations can we perform on an equation that will yield an equivalent equation?

For a clue let us look at Example 2 more closely. Every solution of $3x + 7 = x + 15$ is a solution of $x = 4$ because we added the real number $(-x - 7)$ to both sides of $3x + 7 = x + 15$ and then multiplied both sides by the non-zero real number $\frac{1}{2}$. (Is it certain that $(-x - 7)$ is a real number for every real number x ?)

Now that we know that the truth set of $3x + 7 = x + 15$ is a subset of the truth set of $x = 4$, we are at a crucial point. Do we know that these two equations are equivalent? No, for we have not proved that these truth sets are identical; perhaps the new truth set is larger than the old (remember that in Example 3 we performed an operation which did make the truth set larger).

To be sure that the truth set has not increased, we have to be able to go backwards, too, and show that any solution of the last equation is a solution of the first. What must we do to $x = 4$ to go back to $3x + 7 = x + 15$? You see that every solution of $x = 4$ is a solution of $3x + 7 = x + 15$, because we may multiply both sides of $x = 4$ by the non-zero real number 2 (the reciprocal of $\frac{1}{2}$) and then add to both sides the real number $x + 7$ $\big($the opposite of $(-x - 7)\big)$. Now if every solution of $3x + 7 = x + 15$ is a solution of $x = 4$ and every solution of $x = 4$ is a solution of $3x + 7 = x + 15$, they have identical truth sets; that is, they are equivalent equations.

Exercises 10 - 1b.

1. Using your work on the pairs of equations in Exercises 10 - 1a, list those pairs that are equivalent.
2. Using your work in Problem 1 or in Exercises 10 - 1a, find out if each of the following pairs of equations are equivalent.
   - (a) $4 - 2x = 10$; $x = -3$
   - (b) $12x + 5 = 10 - 3x$; $x = \frac{1}{3}$
   - (c) $x^2 - 4 = 0$; $x = 2$ or $x = -2$
   - (d) $x = 3$; $x(x - 3) = 0$
   - (e) $x - 1 = 0$; $x^2 - 1 = 0$
   - (f) $(x^6 - 4x^5 + x^3 + 1)(x - 1) = 0$; $x^6 - 4x^5 + x^3 + 1 = 0$
   - (g) $|x| = 1$; $x^2 = 1$

3. Using your work in Problem 2, find the truth sets of the following equations:

   (a) $4 - 2x = 10$

   (b) $12x + 5 = 10 - 3x$

   (c) $x^2 - 4 = 0$

   (d) $x(x - 3) = 0$

   (e) $x^2 - 1 = 0$

   (f) $|x| = 1$.

Are you sure in each case you have found the truth set exactly?

4. Often we can simplify one or both sides of a sentence. What kinds of algebraic simplification will yield equivalent sentences? Consider combining terms: Are $3x - 2 - 4x + 6 = 0$ and $-x + 4 = 0$ equivalent? Consider factoring: Are $x^2 - 5x + 6 = 0$ and $(x - 3)(x - 2) = 0$ equivalent? Are $\frac{x^2 - 4}{x - 2} = 4$ and $x + 2 = 4$ equivalent?

In working with the equation $3x + 7 = x + 15$, we added $(-x - 7)$ to both sides. Notice that we could do this with confidence because $(-x - 7)$ is a real number for every value of x . What if we began with the equation $x = 2$ and added $\frac{1}{x - 2}$ to both sides? The new equation

$$x + \frac{1}{x - 2} = 2 + \frac{1}{x - 2}$$

has its truth set empty, whereas the equation $x = 2$ has the truth set $\{2\}$ (recall Problem 9 in Exercises 10 - 1a). In changing our equation we lost the solution 2. The trouble here is that, for some value of x, $\frac{1}{x - 2}$ is not a real number.

Which value of x is the culprit? If we are tempted to add such an expression to both sides of a sentence, or multiply each side by it, we may lose a solution as we did in the present example. In this course we should never perform on a sentence an operation that might lose members of its truth set. Solutions lost by such operations are seldom found again.

In Example 2 we multiplied both sides of the equation by $\frac{1}{2}$. What happens if we multiply both sides of $x - 1 = 0$ by $x - 2$? The new equation $(x - 1)(x - 2) = 0$ has the truth set $\{1, 2\}$, whereas the former has the truth set $\{1\}$. When we choose to multiply both sides of a sentence by an expression, the truth set will not shrink (no solutions will be lost) provided the expression represents a real number for every real number value of the variable. But if the expression has the value 0 for some x , the truth set may get larger, as it did in this example.

Let us use the name real expression to denote an expression which represents a real number for every real number value of the variable. A real number, such as 4 , is automatically a real expression. Why is $(x + 7)$ a real expression? Explain why $\frac{1}{x - 2}$ is not a real expression. Furthermore, let us say that a never-zero real expression is a real expression which is never 0 for any value of the variable. We know that $(x + 7)$ is a real expression, but is it a never-zero real expression? Why? Is $y^2 + 1$ a real expression? A never-zero real expression? How about $1 - |x|$? $\sqrt{t}$? Is the opposite of a real expression also a real expression? Is the reciprocal of a never-zero real expression also a never-zero real expression?

Now we are ready to state the operations on equations which yield equivalent equations:

> If the same real expression is added to both sides of an equation or if both sides are multiplied by the same never-zero real expression, the resulting equation is equivalent to the original.

Now that we know some operations on an equation that will yield equivalent equations, we can state our main procedure for finding the truth set of an equation.

Starting with the equation whose solutions are to be found, build a chain of equations that ends in a simple equation whose truth set is obvious. Be sure that each new equation in the chain is got either by adding the same real expression to both sides of the preceding equation, or by multiplying both sides of the preceding equation by the same never-zero real expression. The truth set of the final simple equation is the truth set of the original equation.

With this method, the truth set of $3x + 7 = x + 15$ can be found like this:

$3x + 7 = x + 15$

$2x = 8$ (by adding the real expression $(-x - 7)$ to both sides)

$x = 4$ (by multiplying both sides by the real expression $\frac{1}{2}$)

The truth set of $x = 4$ is $\{4\}$. Therefore the truth set of $3x + 7 = x + 15$ is $\{4\}$.

As another example, let us see how this works with a more complicated equation.

Example 4. Solve $\frac{x^2}{x^2 + 1} = \frac{1}{2}$.

(We use "solve" to mean "find the truth set of".) Multiplying both sides by the never-zero real expression $2(x^2 + 1)$, we obtain the equivalent equation

$$2x^2 = x^2 + 1 . \qquad \text{(Give reasons.)}$$

Adding the real expression $(-x^2 - 1)$ to both sides of this equation, we have the equivalent equation

$$x^2 - 1 = 0 .$$

This final equation, and hence the original equation, has the truth set $\{1, -1\}$. Why?

Let us recall what happens when we multiply both sides of an equation by a real expression. If the real expression is never-zero, the new equation is equivalent to the old and has the same truth set. If we think (or if we know) that the expression may be 0 for some value of x, the best we can say is that every solution of the first equation is a solution of the second and that the second may have solutions that are not solutions of the first. Nevertheless, this is enough to give us another method for finding the truth set of an equation, as follows. Multiply both sides of the original equation by some real expression, choosing the expression so that the new equation is one whose truth set you can find. Then find the truth set of the new equation. Test each member of the new truth set by substituting it in the original equation. Those that give true numerical sentences form the truth set of the first equation.

Our earlier example, $x + \frac{1}{x - 2} = 2 + \frac{1}{x - 2}$, can be handled this way if we multiply by the real expression $x - 2$. The new equation is $x(x - 2) + 1 = 2(x - 2) + 1$. We have to find the solutions of this equation and test each one in the original equation. The way to find the solutions of the new equation is to follow our main procedure, building a chain of equivalent equations:

$$x(x - 2) + 1 = 2(x - 2) + 1 ,$$

$$x(x - 2) = 2(x - 2) , \qquad \text{(How?)}$$

$$x(x - 2) - 2(x - 2) = 0 , \qquad \text{(Adding what real expression?)}$$

$$(x - 2)(x - 2) = 0 . \qquad \text{(How?)}$$

The last equation obviously has $\{2\}$ for its truth set; so the new equation has 2 for its only solution. When we test 2 in the original equation, the expression $\frac{1}{2 - 2}$ occurs and is not a number. Thus 2 is not a solution of the first equation; hence, the equation has an empty truth set.

Exercises 10 - 1c.

1. Identify each of the following expressions as being one of

these three: Not real, real and sometimes zero, never-zero real. Explain the reasons for your decision.

(a) $x^2 - 4x + 3$

(b) $\frac{3 - 4y}{y + 4}$

(c) $3 + r + \frac{1}{r}$

(d) $\sqrt{t + 1}$

(e) $|y + 1|$

(f) $|y| + 1$

(g) $\frac{x^2}{x^2 + 1}$

(h) $\frac{x^2 + 1}{x^2 + 1}$

(i) $\sqrt{v^2 + 1}$

(j) -3

(k) $\frac{x}{x}$

(l) $\frac{q^2 - 1}{q + 1}$

2. For each of the following pairs of sentences, determine whether or not the pair are equivalent. You can prove a pair are equivalent by beginning with either sentence and applying operations that yield equivalent sentences, until you arrive at the other sentence of the pair. If you think they are not equivalent, try to prove it by finding a number that is in the truth set of one, but not in the truth set of the other.

(a) $2s = 12$; $s = 6$

(b) $5s = 3s + 12$; $2s = 12$

(c) $5s - 4 = 3s + 8$; $s = 6$

(d) $7s - 5s = 12$; $s = 6$

(e) $2x^2 + 4 = 10$; $x^2 = 4$

(f) $3x + 9 - 2x = 7x - 12$; $\frac{7}{3} = x$

(g) $3 + 9x = 4 - 5x$; $x = \frac{1}{5}$

(h) $x^2 = x - 1$; $1 = x - x^2$

(i) $\frac{y - 1}{|y| + 2} = 3$; $y - 1 = 3(|y| + 2)$

(j) $x^2 + 1 = 2x$; $(x - 1)^2 = 0$

(k) $x^2 - 1 = x - 1$; $x + 1 = 1$

(l) $\frac{x^2 + 5}{x^2 + 5} = 0$; $x^2 + 5 = 0$

(m) $v^2 + 1 = 0$; $|v - 1| = -1$

(n) $-1 < t < 3$; $|t - 1| < 2$

3. Change each of the following to a simpler equivalent equation:

(a) $y + 23 = 35$.

(b) $\frac{19}{20}x = 19$.

(c) $6 - t = 7$.

(d) $\frac{1}{7}s = \frac{1}{105}$.

(e) $x(x^2 + 1) = 2x^2 + 2$.

(f) $y(|y| + 1) = |y| + 1$.

4. Solve (that is, find the truth set of), if possible:

(a) $11t + 21 = 32$

(b) $\frac{4}{3} - \frac{y}{5} = \frac{1}{2}$

(c) $\frac{5}{8}x - 17 = 33$

(d) $6 - s = s + 6$

(e) $s - 6 = 6 - s$

(f) $s - 6 = s + 6$

(g) $\frac{y}{3} + \frac{2}{3} = \frac{y}{2} + \frac{3}{2}$

(h) $4x + 3 /2 = x + 6$

(i) $x^4 + x^2 + 1 = x^2$

(j) $y^4+y^3+y^2+y+1=y^4-y^3+y^2-y+1$

(k) $x^2 + 3x = x - \frac{x^2}{2}$

5. Solve:

(a) $\frac{y}{y - 2} = 3$

(b) $\frac{x}{x^2 + 1} = x$

(c) $|x - 1| = 3$

(d) $|x| + 1 = x$

(e) $|8 + 3a| = 7$

6. Show that any pair of numbers (x, y) for which either one of the equations $3x + 18 = y + 23$, $y = 3x - 5$ is true is also a pair for which the other is true. What, then, is the relation between the set of all solution pairs of the first equation and the set of all solution pairs of the second?

7. Show that the equations $4x - \frac{2}{3}y = 6$ and $y = 6x - 9$ are equivalent.

8. Find the dimensions of a rectangle whose perimeter is 30 inches and whose area is 54 square inches.

9. Find three successive integers such that the sum of their squares is 61.

10. For an expression in one variable, the set of values of the variable for which the expression represents a real number is called the domain (or universe) of the expression. What is the domain of a real expression? Determine the domain of each of the expressions of problem 1.

10 - 2. Equivalent sentences: inequalities. If a sentence involves one of the relations "<" , ">" , "≤" , "≥" , we call it an inequality. Let us see what operations may be performed on both sides of an inequality to yield an equivalent inequality.

For example, suppose we try to find the truth set of

$$x - 15 < \frac{x}{4} + 18 .$$

We have available the addition property of order, which states:

For real numbers a, b, c,
$a < b$ if and only if $a + c < b + c$.

This suggests that we may add to both sides of the inequality the real number $(-\frac{x}{4} + 15)$ to obtain $\frac{3}{4}x < 33$. Are you sure that $(-\frac{x}{4} + 15)$ is a real number for every real value of x ? Notice that the opposite, $(\frac{x}{4} - 15)$, is also a real number for every x , and adding $(\frac{x}{4} - 15)$ to both sides of $\frac{3}{4}x < 33$, gives $x - 15 < \frac{x}{4} + 18$. Hence,

$$x - 15 < \frac{x}{4} + 18$$

and

$$\frac{3}{4}x < 33$$

are equivalent inequalities.

Now we recall the multiplication property of order:

For real numbers a, b, c,
if $a < b$ and $c > 0$, then $ac < bc$;
if $a < b$ and $c < 0$, then $ac > bc$.

This property suggests that we multiply both sides of the second inequality by the positive real number $\frac{4}{3}$ to obtain $x < 44$. The reciprocal, $\frac{3}{4}$, is also a positive real number, and multiplying both sides of $x < 44$ by $\frac{3}{4}$ we have $\frac{3}{4}x < 33$. Hence, $\frac{3}{4}x < 33$ and $x < 44$ are equivalent sentences. What can we then say about $x - 15 < \frac{x}{4} + 18$ and $x < 44$? What is the truth set of the original sentence ?

It turns out that the operations we may perform on inequalities to yield equivalent inequalities are somewhat like those for equalities. Let us call a positive real expression one which represents positive real numbers for all real values of the variable. Is $x + 2$ a positive real expression? Is $x^2 + 2$? Is $x^2 - 2$? Is $|x - 2|$?

> If the same real expression is added to both sides of an inequality, the resulting inequality is equivalent to the original.
>
> If both sides of an inequality are multiplied by the same positive real expression, the resulting inequality is equivalent to the original.

Example 1. Solve $\frac{4}{5}y - 6 < \frac{2}{3}y + \frac{5}{6}$.

We may first rid the sentence of fractions by multiplying both sides by the positive real number 30 :

$$24y - 180 < 20y + 25 .$$

Now we add the real expression $-20y + 180$ to both sides:

$$4y < 205 .$$

Finally, we multiply by the positive real number $\frac{1}{4}$:

$$y < \frac{205}{4} .$$

What is the truth set of the original inequality? Explain why all these sentences are equivalent.

Example 2. Solve $\frac{1}{x^2 + 1} < 1$.

Since $x^2 + 1$ is a positive real expression, we may multiply both sides by $x^2 + 1$ to obtain the equivalent sentence

$$1 < x^2 + 1 .$$

By adding -1 to both sides, we have the equivalent sentence

$$0 < x^2 .$$

The truth set of this final sentence is the set of all non-zero reals. This is also the truth set of the original inequality. Explain why.

Exercises 10 - 2.

1. Solve the following inequalities by changing to simpler equivalent inequalities:

   (a) $x + 12 < 39$.

   (b) $\frac{5}{7}x < 35 - x$.

   (c) $\sqrt{2} + 2x > 3\sqrt{2}$.

   (d) $t\sqrt{3} < 3$.

   (e) $8y - 3 > 3y + 7$.

   (f) $\frac{t}{3} < 4 + \frac{t}{6} - 2$.

   (g) $x^2 + 5 \geq 4$.

   (h) $\frac{3}{x^2 + 4} < -2$.

   (i) $\frac{1}{x^2 + 1} \geq 1$.

2. Solve the following sentences:

   (a) $1 < 4x + 1 < 2$.
   (This is equivalent to "$1 < 4x + 1$ and $4x + 1 < 2$") .

   (b) $4t - 4 < 0$ and $1 - 3t < 0$.

   (c) $-1 < 2t < 1$.

   (d) $6t + 3 < 0$ or $6t - 3 > 0$.

   (e) $|x - 1| < 2$.
   (This is equivalent to "$x - 1 < 2$ and $-x + 1 < 2$") .

   (f) $|2t| < 1$.

(g) $|x + 2| < \frac{1}{2}$.

(h) $|y + 2| > 1$.

(This is equivalent to "$y + 2 > 1$ or $-y - 2 > 1$").

(i) $|2x - 4| > 5$.

3. Graph the truth sets of the sentences in problems 2(a) , (c) , (e) and (h) .

4. If a *negative* real expression is one which represents negative real numbers for all real values of the variables, determine which of the following are negative real expressions:

(a) x	(d) $\|-x - 1\|$
(b) $-x$	(e) $-\|x + 1\|$
(c) $\frac{1}{-x^2 - 1}$	(f) -5

5. Suppose we multiply both sides of an inequality by the same negative real expression. What do we have to do to the *order of the inequality* if the resulting inequality is to be true? Are the two inequalities then equivalent?

Solve the following:

(a) $-3x > 4$; (b) $-(x - 1) < 2$; (c) $-(2x - 4) > 5$.

6. Solve $3y - x + 7 < 0$ for y ; that is, obtain an equivalent sentence with y alone on the left side. What is the truth set for y when $x = 1$? Now solve for x . What is the truth set for x if $y = -2$?

10 - 3. *Factored equations*. When in Chapter 9 you solved factored quadratic equations such as

$$(x - 3)(x + 2) = 0 ,$$

you needed the important property of numbers:

> For real numbers a and b ,
>
> $ab = 0$ if and only if $a = 0$ or $b = 0$.

Restate this property for the particular a and b in the above equation. Interpret the "if and only if" in your own words. It is this basic property and the fact that $x - 3$ and $x + 2$ are real numbers for every real number x , that guarantee the truth set of the sentence "$x - 3 = 0$ or $x + 2 = 0$", namely, $\{3, -2\}$. In other words,

> The equation "$ab = 0$" is equivalent to the sentence "$a = 0$ or $b = 0$", provided a and b are real expressions.

How would you extend this property to equations such as $abcd = 0$? State a general property for any number of factors. What is the truth set of

$$(x + 1)(x - 3)(2x + 3)(3x - 2) = 0 ?$$

The union of two sets A and B is the set of numbers that are either in A or in B (or in both). Thus the union of $\{3,2\}$ and $\{3,1,-1\}$ is $\{3,2,1,-1\}$. What is the union of $\{5,4,2,0\}$ and $\{6,4,1,0\}$? What is the relationship between the truth set of "$(x - 3)(x - 2)(x + 2) = 0$" and the truth sets of "$(x - 3)(x - 2) = 0$" and "$x + 2 = 0$" ? This example illustrates the fact that when a and b are thought of as being expressions such as $(x - 3)(x + 2)$ and $(x - 2)$, or even more complicated ones, the above property can be restated as follows:

The truth set of the equation $ab = 0$ is the union of the truth sets of the two equations $a = 0$, $b = 0$, when a and b are real expressions.

Exercises 10 - 3a.

1. Find the truth sets of:
   (a) $(x + 2)(x + 1)(x - 2)(x + 3) = 0$.
   (b) $(3x - 1)(2x + 1)(x - 2)(4x - 3) = 0$.
2. Solve:
   (a) $(x^2 - x + 2)(x + 3) = 0$
   (b) $(x^2 - 1)(x^2 + 5x + 6) = 0$
   (c) $(x^2 - 5)(x^2 - 2) = 0$
   (d) $x(3x^2 - 3x)(x^2 - 4) = 0$
   (e) $x^3 + x = 2x^2$
   (f) $y^2 + 2 = 0$
3. Solve $x^3 = 1$ by arguing about the "size" of x.
   (If $x < 1$, what about x^3? If $x > 1$, what about x^3?)
4. Solve $x^4 = 1$ by writing it $(x^2)^2 - 1 = 0$ and factoring.
5. Find a polynomial which has the value 0 each time x takes a value in the set $\{\sqrt{2}, -\sqrt{2}, 1\}$.
6. Find the truth set of the sentence

$$(x - 3)(x - 1)(x + 1) = 0 \text{ and } |x - 2| < 2 .$$

Now we can learn more about the case in which we multiply both sides of an equation by an expression which for some value of the variable may be 0 or not a real number.

Consider this example: Solve

$$(x - 3)(x^2 - 1) = 4(x^2 - 1) .$$

Our first impulse is to multiply both sides by $\frac{1}{x^2 - 1}$. But $\frac{1}{x^2 - 1}$ is not a real expression. Why? Instead, since $4(x^2 - 1)$ is a real expression, let us add $-4(x^2 - 1)$ to both sides, giving

$$(x - 3)(x^2 - 1) - 4(x^2 - 1) = 0 ,$$

$$(x - 3 - 4)(x^2 - 1) = 0 , \qquad \text{(Why?)}$$

$$(x - 7)(x - 1)(x + 1) = 0 .$$

Among these sentences each is equivalent to every other. What is the truth set of the original sentence? If we had multiplied each side (unthinkingly) by $\frac{1}{x^2 - 1}$, what would by the truth set of the resulting sentence?

We see that, in general, the following are equivalent sentences when a, b, and c are any real expressions:

$$ac = bc ,$$

$$ac - bc = 0 ,$$

$$(a - b)c = 0 ,$$

$$a - b = 0 \text{ or } c = 0 .$$

This tells us that the truth set of the sentence

$$ac = bc$$

is the union of the truth sets of $a - b = 0$, $c = 0$, when a, b, and c are real expressions.

Exercises 10 - 3b.

1. Multiply both sides of the equation "$x^2 = 3$" by $(x - 1)$. Compare the new and the original truth sets.

2. Multiply both sides of the equation "$t^2 = 1$" by $(t + 1)$. Compare the new and the original truth sets. Discuss any differences the two multiplications made in the truth sets in problems 1 and 2.

3. Solve:

   (a) $(3 + x)(x^2 + 1) = 5(3 + x)$.

   (b) $(2 + x)(x^2 + 1) = 5(2 + x)$.

   (c) $3(x^2 - 4) = (4x + 3)(x^2 - 4)$.

   (d) $(4x + 3)(x^2 - 4) = 11(x^2 - 4)$.

10 - 4. Fractional equations. The expression $\frac{1}{x}$ is not a real number when x is 0. Therefore, when we try to solve the equation

$$\frac{1}{x} = 2$$

we are limited to numbers other than 0. Knowing that x cannot be 0, we may then multiply by the non-zero number x to obtain

$$\frac{1}{x} \cdot x = 2x ,$$

$$1 = 2x .$$

Hence, "$\frac{1}{x} = 2$" and "$1 = 2x$ and $x \neq 0$" are equivalent sentences. The latter has the truth set $\{\frac{1}{2}\}$. Thus $\frac{1}{2}$ is the solution.

Another way to handle this same problem is to add -2 to both sides of $\frac{1}{x} = 2$, giving

$$\frac{1}{x} - 2 = 0 ,$$

$$\frac{1 - 2x}{x} = 0 . \qquad \text{(Why?)}$$

What are the requirements on a and c for the number

$\frac{a}{c}$ to be 0 ? They are first, that $c \neq 0$ (why?) , and second, that $a = 0$ (why?) . Thus the sentence "$\frac{a}{c} = 0$" is equivalent to the sentence "$a = 0$ and $c \neq 0$".

Then $\frac{1 - 2x}{x} = 0$ is equivalent to what sentence? Your answer should be "$1 - 2x = 0$ and $x \neq 0$", which is the same sentence we had before. Can you find the truth set of $\frac{x + 1}{x - 2} = 0$?

The same two approaches can be used on more complicated fractional equations. Thus we can solve the equation

$$\frac{1}{x} = \frac{1}{1 - x}$$

either by multiplying both sides by a suitable polynomial (what is it?), or by writing it first as $\frac{1}{x} - \frac{1}{1 - x} = 0$ and then simplifying to a single fraction. In either case we must recognize two "illegal values" for x. What are they? And the solution is subject to x not taking on those values. Using the second method, we get $\frac{(1 - x) - x}{x(1 - x)} = 0$ which is equivalent to $1 - 2x = 0$, $x \neq 0$ and $x \neq 1$. The solution of this sentence is $\frac{1}{2}$, which is, therefore, the solution of the original sentence.

As a final example, solve

$$\frac{x}{x - 2} = \frac{2}{x - 2} .$$

If $x \neq 2$, then upon multiplying both sides by $x - 2$ we get

$$\frac{x}{x - 2} \cdot (x - 2) = \frac{2}{x - 2} \cdot (x - 2) ,$$
$$x = 2 .$$

Hence the sentence "$\frac{x}{x - 2} = \frac{2}{x - 2}$" is equivalent to the sentence "$x \neq 2$ and $x = 2$". What is the truth set of this sentence?

Exercises 10 - 4.

Solve the following equations:

1. $\frac{2}{x} - \frac{3}{x} = 10$

2. $\frac{x}{2} - \frac{x}{3} = 10$

3. $\frac{3}{2y} - \frac{2 + 5y}{y} = \frac{1}{3}$

4. $\frac{1}{t} = \frac{1}{t - 1}$

5. $x + \frac{1}{x} = 2$

6. $y - \frac{2}{y} = 1$

7. $\frac{1}{y} - \frac{1}{y - 4} = 1$

8. $\frac{t}{1 + t} + \frac{1}{t - 1} = 0$

9. $\frac{s - 2}{s} + \frac{3}{s^2} = 1$

10. $\frac{1 - y}{1 + y} + \frac{1 + y}{1 - y} = 0$

11. $\frac{1 - y}{1 + y} - \frac{1 + y}{1 - y} = 0$

12. $\frac{1}{x} + \frac{1}{1 - x} + \frac{1}{1 + x} = 0$

13. $\frac{1}{y} + \frac{2}{1 - y} + \frac{1}{1 + y} = 0$

14. $(\frac{x - 1}{x + 1})^2 = 4$

15. $\frac{-2}{x - 2} + \frac{x}{x - 2} = 1$

16. $(\frac{x}{x + 1})(x^2 - 1) = 0.$

10 - 5. Squaring. If $a = b$, then of course $a^2 = b^2$. Why? Do you think it is true, conversely, that if $a^2 = b^2$, then $a = b$? You may see at once that this is not so. Give an example. In any case we can alter $a^2 = b^2$ through a chain of equivalent sentences as follows:

$$a^2 = b^2 ,$$

$$a^2 - b^2 = 0 ,$$

$$(a - b)(a + b) = 0 ,$$

$$a - b = 0 \text{ or } a + b = 0 ,$$

$$a = b \text{ or } a = -b .$$

Tell why each of these sentences is equivalent to the next one.

We finally see that "$a^2 = b^2$" is not equivalent to "$a = b$", but rather to "$a = b$ or $a = -b$".

For example, if we square both sides of the sentence "$x = 3$" we obtain "$x^2 = 9$". Then $x^2 - 9 = 0$; $(x + 3)(x - 3) = 0$; $x = 3$ or $x = -3$.

Exercises 10 - 5a.

See what squaring both sides does to the truth sets of the following equations. In each case the factoring will be of the type $a^2 - b^2 = (a - b)(a + b)$.

1. $x = 2$.
2. $(x - 1) = 1$.
3. $x + 2 = 0$.
4. $x - 1 = 2$.

In these cases it is obvious what the original truth set is, and we haven't had to use the new truth set to obtain the old. However, sometimes we square both sides of an equation as a simplifying process in situations where we don't already know the truth set. We do know, as in the above exercises, that any solution of the unsquared equation is a solution of the squared equation. But we also know that the new truth set may be larger than the old. Therefore each solution of the squared equation must be checked in the original equation in order to eliminate any possible extra solutions that may have crept in during the squaring.

Example 1. Solve $\sqrt{x} + x = 2$.

It is clear that if we square both sides as the equation stands we shall still have a square root left in the left side. We therefore first go to the equivalent equation "$\sqrt{x} = 2 - x$", and now square both sides obtaining

$$x = 4 - 4x + x^2 ,$$

$$x^2 - 5x + 4 = 0 ,$$

$$(x - 4)(x - 1) = 0 ,$$

$$x = 4 \quad \text{or} \quad x = 1 .$$

Checking back, we see that 4 does not make the original equation true, while 1 does. The solution is therefore 1. (4 is the solution of the other alternative sentence corresponding to the second possibility, $a = -b$. Do you see what this equation is? Answer: $\sqrt{x} = -(2 - x)$.)

Example 2. Solve $|x| - x = 1$.

We use the fact that $|x|^2 = x^2$ for every real number x to get the sequence of sentences

$$|x| = x + 1 ,$$

$$x^2 = x^2 + 2x + 1 ,$$

$$2x + 1 = 0 ,$$

$$x = -\frac{1}{2} .$$

Checking back, we find that $-\frac{1}{2}$ does make the original equation true and is, therefore, its solution.

Exercises 10 - 5b.

Solve the following equations by squaring:

1. $x - |x| = 1$.

2. $|2x| = x + 1$.
3. $2x = |x| + 1$.
4. $x = |2x| + 1$.
5. $\sqrt{2x} = 1 + x$.
6. $\sqrt{2x + 1} = x + 1$.
7. $\sqrt{x + 1} - 1 = x$.
8. $\sqrt{4x} - x + 3 = 0$.
9. $\sqrt{x + 4} = x - 2$.
10. $|x - 2| = 3$.
11. The time t in seconds it takes a body to fall from rest a distance of s feet is given by the formula $t = \sqrt{\frac{2s}{g}}$. Find s if $t = 6.25$ seconds and $g = 32$.
12. Determine whether the following pairs of sentences in two variables are equivalent:
    (a) $x^2 + y^2 = 1$, $y = \sqrt{1 - x^2}$.
    (b) $x^2 + y^2 = 1$, $\sqrt{x^2 + y^2} = 1$.
    (c) $x^2 = xy$, $x = 0$ or $x = y$.

10 - 6. Polynomial inequalities. (Optional) Is $(-4)(3)(5)(-6)(-8)$ a positive number? A negative number? Did you need to perform the multiplication to answer this question?

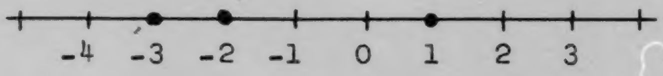
When we multiply several non-zero numbers together, their product is positive if the number of negative factors is even, and their product is negative if the number of negative factors is odd.

This means that we can tell immediately whether a factored polynomial, such as

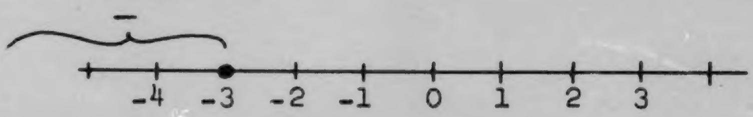
$$(x + 3)(x + 2)(x - 1) ,$$

is positive, negative, or 0 for any given x . How about this polynomial for $x = 2$? For $x = 0$? For $x = -1$? For $x = -\frac{5}{2}$? For $x = -4$? You need not compute the value of the polynomial; just check how many factors are negative.

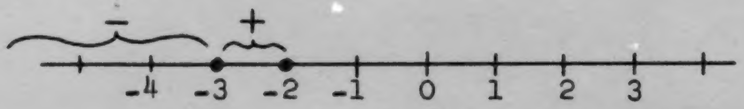
But we can do better than choosing a few points at random. We can first find the set of values of x for which $(x + 3)(x + 2)(x - 1)$ is 0 (the truth set of $(x + 3)(x + 2)(x - 1) = 0$) . What is this set? Then we graph this set on the number line.



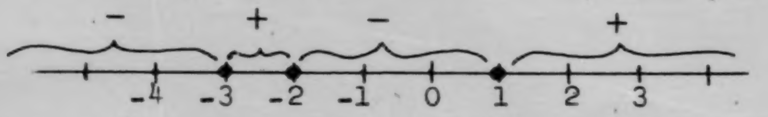
Now what can we say about each of the factors $(x + 3)$, $(x + 2)$, $(x - 1)$ for any x less than -3 ? Try $x = -4$. We find that all three factors are negative numbers, and therefore their product is negative. We indicate this on the number line as follows:



What about these factors when x is between -3 and -2 ? Try $x = -\frac{5}{2}$. The factor $(x + 3)$ is now positive, while the other two remain negative. We can think of $(x + 3)$ as "changing over " from negative to positive as x crosses -3 . The product is now positive for x between -3 and -2 .



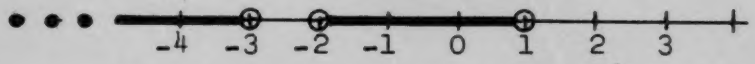
Probably you can see what is going to happen when x crosses -2 and finally 1. When x crosses -2, the next factor $(x + 2)$ changes from negative to positive, so that for any x between -2 and 1 there are two positive factors and one negative, the total product now being negative. Finally, when x crosses 1, the last factor changes from negative to positive, so that for x greater than 1, all factors are positive.



Using this final diagram, we can read off the truth sets of certain related inequalities. For example, the truth set of the sentence

$$(x + 3)(x + 2)(x - 1) < 0$$

is graphed below. This is the set of all numbers x for



which the product of the factors is negative, namely, the set of all x such that

$$x < -3 \text{ or } -2 < x < 1 .$$

What is the truth set of the sentence $(x + 3)(x + 2)(x - 1) > 0$? Graph it. Of the sentence $(x + 3)(x + 2)(x - 1) \geq 0$?

To find the truth set of

$$x^2 - 3 \leq 2x ,$$

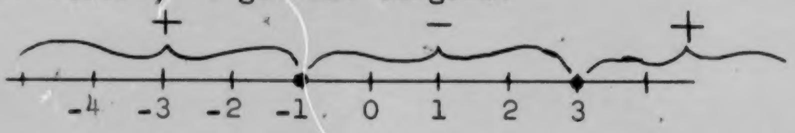
we first change to the equivalent inequality with 0 on the right side:

$$x^2 - 2x - 3 \leq 0 .$$

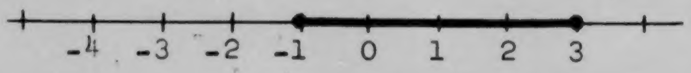
Then we factor the left side into first degree factors:

$$(x + 1)(x - 3) \leq 0 .$$

Proceeding as before, we get the diagram:



Thus, the truth set of the inequality $x^2 - 3 \leq 2x$ has the following graph (since the product of the factors $(x + 1)(x - 3)$ must be negative or zero).



This is the set of all x such that $-1 \leq x \leq 3$.

Exercises 10 - 6a.

1. Using the above discussion as a model, graph and describe the truth sets of the following inequalities:

   (a) $(x - 1)(x + 2) > 0$

   (b) $y^2 < 1$

   (c) $t^2 + 5t \leq 6$

   (d) $x^2 + 2 \geq 3x$

   (e) $(s + 5)(s + 4)(s + 2)(s)(s - 3) < 0$

   (f) $2 - x^2 < x$

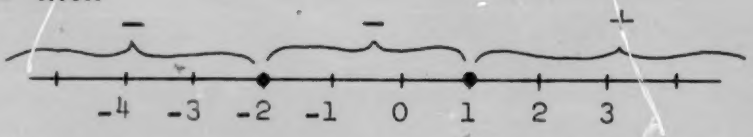
2. What is the truth set of the sentence

   $$(x + 2)(x - 1) > 0 \text{ and } x < 3 ?$$

3. Is there a single polynomial inequality equivalent to the sentence of Problem 2 ?

There is one danger point which we should explain.

Suppose a factor is repeated in a polynomial one or more times, as in $(x + 2)^2(x - 1)$. When x crosses -2, there are two factors of the polynomial changing together from negative to positive. The number of negative factors drop from 3 to 1, and the product remains negative as x crosses -2. The diagram is then



What is the truth set of

$$(x + 2)^2(x - 1) > 0 ?$$

(i.e. , for what values of x is the product of the factors positive?) Of

$$(x + 2)^2(x - 1) \leq 0 ?$$

What happens if a factor occurs three times, as in $x(x - 1)^3$? What is the truth set of

$$x(x - 1)^3 < 0 ?$$

Of

$$x(x - 1)^3 \geq 0 ?$$

Sometimes we have a quadratic factor, such as $x^2 + 2$, which cannot be factored and which is always positive for all values of x. Does such a factor have any effect on the way the product changes from positive to negative? What is the truth set of

$$(x^2 + 2)(x - 3) < 0 ?$$

Of

$$(x^2 + 2)(x - 3) \geq 0 ?$$

Exercises 10 - 6b.

1. Solve and graph:

   (a) $x^2 + 1 > 2x$.

   (b) $x^2 + 1 < 0$.

   (c) $(t^2 + 1)(t^2 - 1) \geq 0$.

   (d) $4s - s^2 > 4$.

   (e) $(x - 1)^2(x - 2)^2 > 0$.

   (f) $(y^2 - 7y + 6) \leq 0$.

   (g) $(x + 2)(x^2 + 3x + 2) < 0$.

   (h) $3y + 12 \leq y^2 - 16$.

   (i) $x^2 + 5x > 24$.

   (j) $|x|(x - 2)(x + 4) < 0$.

Chapter 11

Graphs of Open Sentences in Two Variables

11 - 1. The real number plane. The real number line has been a help in making decisions about relations among real numbers. (Give examples of some cases in which we have used the real number line.) Perhaps a real number plane would be even more helpful.

We have associated numbers with points of the line. How can we associate numbers with points of the plane? Consider any point P in a plane. If this point is on the number line x units from the zero point, then there is one number x associated with P. If P is not on the number line, as in Figure 1, there is a number, in this case 7, which is associated with P in the sense that it is on the number line directly under P (it would be directly over P if P were below the number line). This number is not sufficient by itself to locate P, since 7 is on the number line

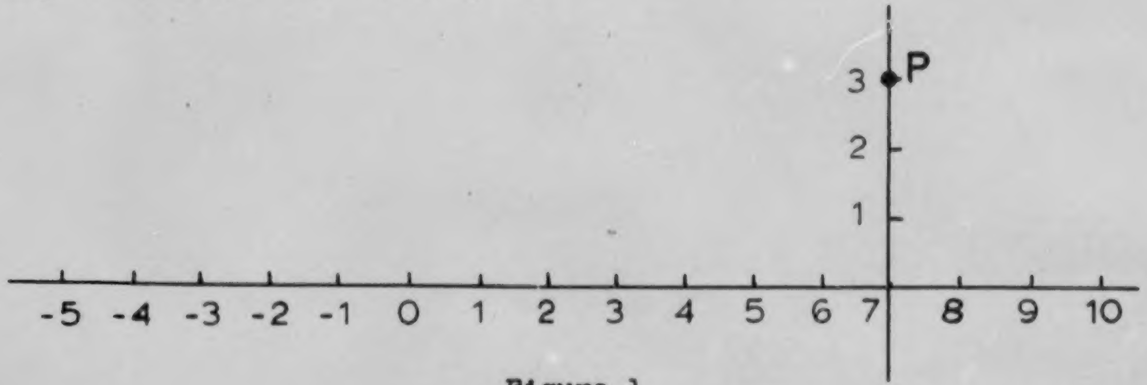


Figure 1.

and p is 3 units above the line. Perhaps we need two numbers to name P, using 7 as the first one because it is on the number line which we already know. From 7 we could draw a second number line to P, drawing it vertical and with the zero of this line coinciding with 7 on the first line. On this vertical line we find p at the number 3; this number we regard as the second number belonging to P. The point P now has the associated pair of numbers (7, 3). We write these as indicated, placing the number found along the horizontal line first, and the one found along the vertical line second, and enclosing them in parentheses. We have now assigned to p a first number, 7, and a second number, 3, and we think of these as an ordered pair of numbers, (7, 3), belonging to p and called the coordinates of P. The first number is called the abscissa of p and the second number the ordinate of P.

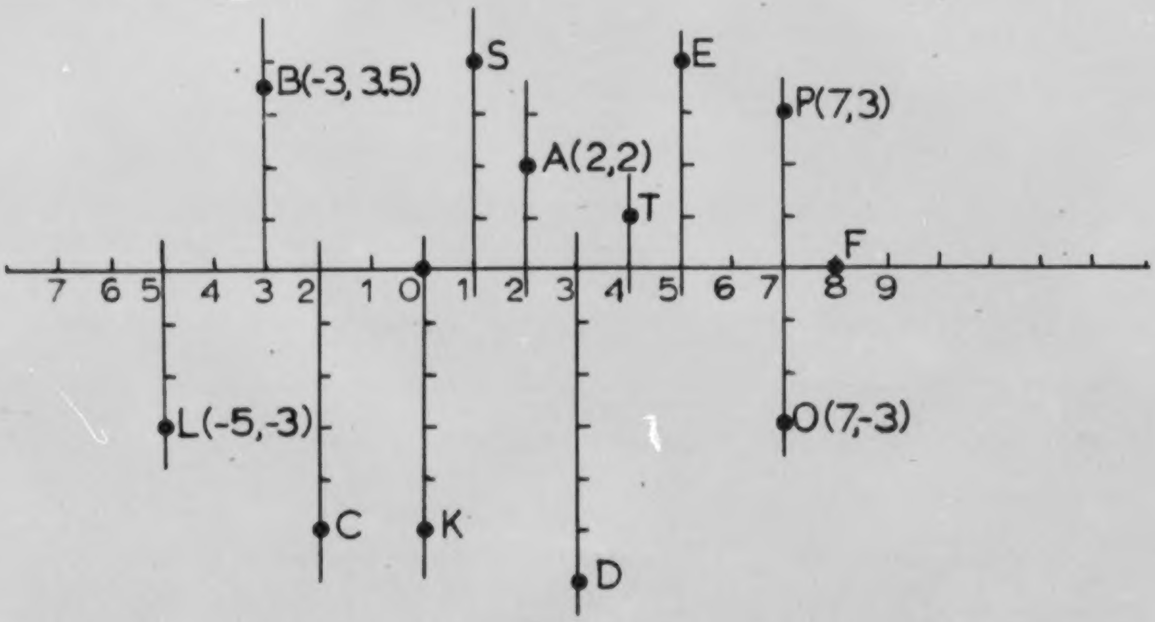


Figure 2.

In Figure 2, verify that the ordered number pairs written for point P, A, B, L, and Q are correct. Which number is written first in each case? Which is second? The order is important, and hence we consider only ordered pairs. Note that the ordered pair (1, 4) is not the same as the ordered pair (4, 1). How does the ordered pair for Q differ from that for P ? Why is the second number for Q negative?

In Figure 2, what ordered pairs of numbers are associated with points E, C, K, and D? What ordered pair is associated with point H, which is on the number line? This point is called the origin, and is associated with (0, 0). We may consider that the second number of the ordered pair is the distance above or below the number line. What ordered pairs are associated with points F and G ? Make a general statement about the second number of the ordered pair associated with any point which lies on the horizontal number line.

If we have several points in a plane, and a single horizontal number line, is there any way in which we can refine our figure so that we can identify the ordered pairs of numbers associated with the points without drawing a separate vertical line from each point to the horizontal number line? For this purpose we shall use coordinate paper and draw only one second number line, passing through the origin and perpendicular to the first number line. If we label the units of measure on both of these lines, each of which is called a coordinate axis, the network of the coordinate paper will permit us to choose quickly the suitable numbers of an ordered pair. In fact, labeling the units may be omitted if one square is used to represent one unit on each axis. We shall adopt this practice throughout the discussion in this chapter.

Note that S and T in Figure 2 do not have the same coordinates; the first coordinate of S is 1 but the first coordinate of T is 4. The coordinates of S are the ordered number piar (1, 4), while those of T are the ordered number pair (4, 1). The same numbers appear in each pair, but since the order is different, the ordered pairs are different.

Do you think it is always true that two different points in the plane cannot have the same coordinates?

Exercises 11 - 1a.

1. Write the ordered pairs of numbers which are associated with the points A through M in the figure below:

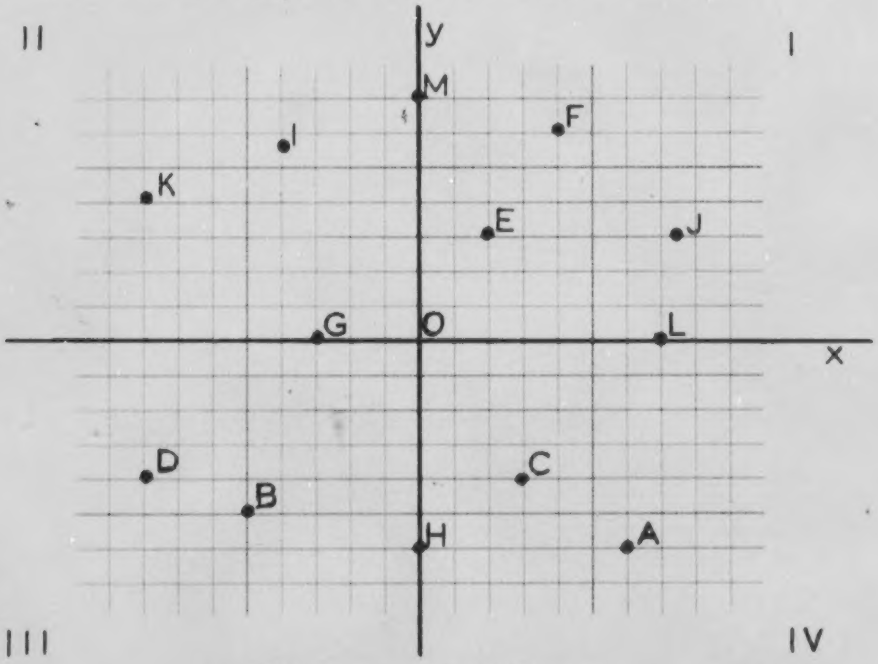


Figure for Problem 1.

2. The four parts of the plane into which the number lines divide the plane are called quadrants. These quadrants are numbered counter-clockwise, as in the figure for problem 1 beginning with the quadrant in the upper right-hand corner where both coordinates are positive. In which quadrants will the points lie for which the second coordinate is equal to the first?
3. Where are all the points whose ordered pairs have ordinates -3 ? Locate as many of these as possible. This set of points forms what sort of figure?
4. Where are all the points whose ordered pairs have abscissas $\frac{3}{2}$? Locate as many as possible.

Now let us shift our attention from points to numbers. Select any real number as the first number of a pair, and then select any real number as the second of the pair. Can we associate with each such ordered pair a point on the plane? Given the ordered pair $(-3, \frac{2}{3})$, can you find a point of the plane having these coordinates? Explain how you locate the point. Does every ordered pair correspond to at least one point? Exactly one point?

Is it clear that for each point of the plane there is exactly one ordered pair of real numbers and for each ordered pair there is exactly one point? This is an important fact in mathematics.

Exercises 11 - 1b.

1. Using coordinate axes of your own choosing, locate the points associated with the following ordered pairs of numbers:

    $A(1, -3)$
    $B(-6, 4)$
    $C(0, \frac{8}{3})$
    $D(-7, -1)$
    $E(-4, 0)$
    $F(0, 0)$
    $G(5, \frac{3}{2})$
    $H(\frac{3}{2}, 5)$
    $I(-4, -6)$
    $J(-6, -4)$
    $K(0, -\frac{5}{3})$
    $L(-\frac{5}{3}, 0)$

2. In the figure you have just drawn for problem 1, are points G and H the same? Why? Are points I and J the same? K and L ?

3. With reference to a set of coordinate axes, mark the points with coordinates: $(2, 3)$, $(2, 1)$, $(2, \frac{1}{2})$, $(2, 0)$, $(2, -5.5)$, $(2, -\frac{7}{2})$. What is true of all of these ordered pairs of numbers? Where are all the points for which the abscissa of the ordered pair is 2 ?

4. If you locate several points whose ordered pairs have 5 for their ordinates, where would all these points lie?

5. With reference to a set of coordinate axes, locate eight points whose coordinates are pairs of numbers for which the first and second numbers are the same. If you could locate all such points, what sort of figure would you have?

6. Let us think of moving all the points of a plane in the following manner: Each point with coordinates (a, b) is moved to the point with coordinates $(-a, b)$. Describe this in terms of taking the opposite. Another way of looking at

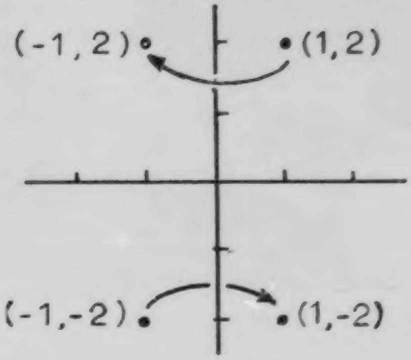


Figure for Problem 6.

this is to consider that the points of the plane are rotated one-half a revolution about the y-axis, as indicated in the figure for problem 6. Answer the following questions, and locate the points referred to in parts (a) and (b):

(a) Into what points do the following points go: (2, 1), (2, -1), $(-\frac{1}{2}, 2)$, (-1, -1), (3, 0), (-5, 0), (0, 2), (0, -2) ?

(b) What points go into the points listed in (a) above?

(c) What point does (a, -b) go into?

(d) What point does (-a, b) go into?

(e) What point goes into (a, b) ?

(f) What points go into themselves?

7. Suppose a point with coordinates (a, b) is moved into the point (a + 2, b). This can be thought of as sliding the points of the plane to the right 2 units. Answer the following questions and locate all of the points in parts (a) and (b).

(a) What points do the following points go into: (1, 1), (-1, -1), (-2, 2), (0, -3), (3, 0) ?

(b) What points go into the points listed in (a) above?

(c) Into what point does (a - 2, b) go?

(d) What point goes into (-a, b) ?

(e) Which points go into themselves?

11 - 2. Graphs of open sentences with two variables.

If we assign the values 0 and -2 to the variables of the open sentence

$$3y - 2x + 6 = 0 ,$$

is it then a true sentence? To which variable did you assign 0 ? To which -2 ? Were there two different ways to make these assignments?

To avoid the kind of confusion you met in the preceding paragraph, let us agree that whenever we write an open sentence with two variables, we must indicate which of the variables is to be taken first. When the variables are x and y, as in the above example, x will always be taken first.

With this agreement we are ready to examine the connection between an open sentence with two ordered variables and an ordered pair of real numbers. Among the set of all ordered pairs of real numbers, each pair has a first number which we associate with the first variable of the sentence and a second number which we associate with the second variable. In this way an open sentence with two ordered variables acts as a sorter--it sorts the set of all ordered pairs of real numbers into two subsets:

(1) the set of ordered pairs which make the sentence true, and

(2) the set of ordered pairs which make the sentence false.

As before, we call this first set the truth set of the sentence.

Now we can answer the question in the first paragraph if we specify the ordered pair $(0, -2)$. Does the ordered pair $(0, -2)$ belong to the set of ordered pairs for which

$$3y - 2x + 6 = 0$$

is a true sentence? Does the ordered pair $(-2, 0)$ belong to the truth set?

An ordered pair belonging to the truth set of a sentence with two variables is called a solution of the sentence, and this ordered pair is said to satisfy the sentence. If r is taken as the first variable, what are some solutions of

$$s = r + 1 \ ?$$

Does the ordered pair $(-2, -3)$ satisfy this sentence? Is $(-3, -2)$ a solution?

If u is taken as the first variable, what are some ordered pairs in the truth set of

$$v = 2u^2 \ ?$$

Is $(-1, 2)$ a solution of this sentence? Does $(2, -1)$ satisfy this sentence?

Throughout this chapter we will use only x and y as variables, in order to focus attention on properties of sentences with two ordered variables. But many times in the future you will see other variables used, and then you must always decide which variable is used first.

One other point needs to be stressed. The sentence "$y = 4$" can be considered as a sentence in one variable y, or it can be considered as a sentence with two ordered variables x and y. When we say that "$y = 4$" is a sentence with two variables, we mean that "$y = 4$" is an abbreviation for

$$(0)x + (1)y = 4 \ .$$

What are some solutions of this sentence? What is true of every ordered pair satisfying this sentence? If "$x = -2$" is a sentence with two variables, then "$x = -2$" is an abbreviation for

$$(1)x = (0)y = -2 .$$

What is true of every ordered pair satisfying this sentence?

Exercises 11 - 2a.

1. Describe the truth sets of the following open sentences in two ordered variables x, y:
   - (a) $y = 5$
   - (b) $x = 0$
   - (c) $y = -3x$
   - (d) $x = 3$
2. Find four solutions of each of the following open sentences:
   - (a) $y = 3x - 2$
   - (b) $y = 2 + x$
   - (c) $y = x^2 + 1$
   - (d) $y = |x|$
3. For each of the sentences in Problem 2, find two ordered pairs which do not satisfy the sentence.
4. With respect to separate sets of coordinate axes, locate the points whose coordinates are the solutions you found for each of the sentences in Problem 2.

Remember that every point of the plane has an associated pair of numbers called its coordinates. Now we see that an open sentence with two ordered variables not only sorts the set of ordered pairs of numbers into two subsets--it also sorts the points of the plane into two subsets:

(1) the set of all points whose coordinates satisfy the sentence, and

(2) all other points.

As before, we call this first set of points the graph of the sentence.

We will be interested to learn what sort of figure on the plane this graph will be for any given sentence. Let us try, as an example, the sentence

$$2x - 3y - 6 = 0 .$$

We can guess several solutions, such as (3, 0) and (0, -2). Try to guess some more solutions. Notice that it would be easier to determine solutions if we write an equivalent equation having y by itself on the left side:

$$\begin{aligned} 2x - 3y - 6 &= 0 , \\ -3y &= -2x + 6 , \\ 3y &= 2x - 6 , \\ y &= \frac{2}{3}x - 2 . \end{aligned}$$

We call this last equivalent sentence the y-form of the original sentence. Now we see that "$y = \frac{2}{3}x - 2$" can be translated into an English sentence in terms of abscissas and ordinates of points on its graph: "The ordinate is 2 less than $\frac{2}{3}$ of the abscissa."

Since we shall be taking $\frac{2}{3}$ of the abscissa, it might be simpler to use abscissas which are multiples of 3. If the abscissa is 3, the ordinate must be 0 in order to form a solution. Why? If the abscissa is -6, what must the ordinate be? Continuing, we can fill in a table of ordered pairs which satisfy the sentence:

x	-9	-6	-3	0		5	
y		-6		-2	0		9

You fill in the empty squares.

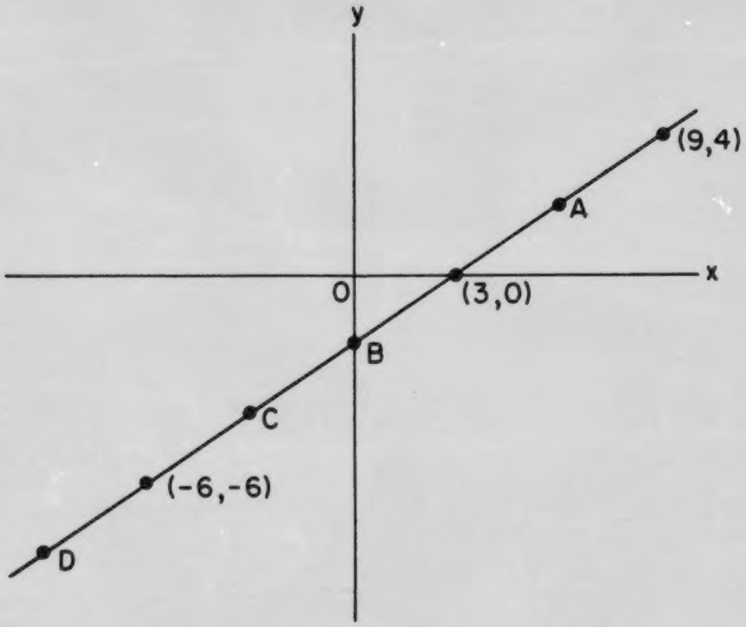


Figure 3.

In Figure 3, the points (-6, -6), (3, 0) and (9, 4) seem to lie on a straight line. Do the other points whose coordinates you found in the table of solutions also seem to lie on this line? This brings up the question: If we draw the line through these points, will we find on it all the points such that each has "ordinate 2 less than $\frac{2}{3}$ of the abscissa?" Furthermore, we must ask: Is every point on this line a point whose "ordinate is 2 less than $\frac{2}{3}$ of the abscissa" ?

Suppose we try a point which appears to be on the line, such as point A in Figure 3. The coordinates of this point are (6, 2). Do these coordinates satisfy the equation $2x - 3y - 6 = 0$?

When we say that a specified line is the graph of a particular open sentence, we mean that both our questions above are answered affirmatively:

(1) if two ordered numbers satisfy the sentence, they are the coordinates of a point on the line;

(2) if a point is on the line, its coordinates satisfy the open sentence.

Thus, the line drawn in Figure 3 is the graph of the sentence

$$2x - 3y - 6 = 0 .$$

We can do the same with such open sentences as $3y + 5x - 11 = 0$, $2x + 5 = 0$, $-8y + 1 = 0$, etc., and in each case reach the same conclusion:

> There is a line which is the graph of the sentence. In fact, given any open sentence of the form
>
> $$Ax + By + C = 0,$$
>
> where A, B, and C are real numbers with A and B not both 0, there is a line that is the graph of the open sentence.

Conversely, given any line in the plane, there is an open sentence of the above form that has the line as its graph.

Exercises 11 - 2b.

1. Where are all the points in the plane whose ordinates are -3 ? Locate as many of these as possible.

2. With reference to one set of coordinate axes, locate the set of points whose coordinates satisfy the following equations, that is, the points whose pairs of coordinates belong to the truth sets of the equations.

   (a) $y = 5$ (c) $x = 0$

   (b) $y = 0$ (d) $x = -2$

   What is the equation whose graph is the horizontal axis?

   What is the equation whose graph is the vertical axis?

3. With reference to a set of coordinate axes, find the points such that each has the abscissa equal to the opposite of the ordinate, using all possible pairs of real numbers which have meaning within the scope of your graph. With reference to the same axes, locate the points such that each has ordinate twice the abscissa; the points such that each has ordinate that is the opposite of twice the abscissa. What general statements can you make concerning these graphs? Write open sentences for each of the graphs drawn.

4. With reference to one set of coordinate axes, draw the graphs of the following:

   (a) $y = 3x$ (d) $y = -3x$

   (b) $y = 6x$ (e) $y = -6x$

   (c) $y = \frac{1}{2}x$ (f) $y = -\frac{1}{2}x$

What characteristic do all of these have in common? How does the graph of (a) differ from the graph of (d) ? Does the same pattern apply to the graphs of (b) and (e) ? To the graphs of (c) and (f) ?

5. With reference to one set of coordinate axes, draw the graphs of the following, and label each one:

(a) $y = x + 5$ (d) $y = 2x - 5$

(b) $y = x - 3$ (e) $y = \frac{1}{3}x + 2$

(c) $y = 2x + 5$ (f) $y = -\frac{1}{3}x - 2$

How does the graph of (a) differ from the graph of (b) ? How does the graph of (c) differ from the graph of (d) ? How does the graph of (e) differ from the graph of (f) ? What is true of the graphs of (a) and (b), and also of the graphs of (c) and (d), that is not true of the graphs of (e) and (f) ?

6. With reference to a set of coordinate axes, find five points for which the ordinate is greater than the abscissa. Are all of these points on one line? Where are they? What is the sentence whose graph is this set of points? Where are the points whose abscissa is less than the ordinate? What is the sentence whose graph is this set of points? When we wish to draw the graphs of sentences such as these, we shall indicate the graphs by shading in the portion of the plane which contains those points whose coordinates make the sentence true, as in Figures 4 and 5. If the verb is "is greater than or equal to", or "is less than or equal to",

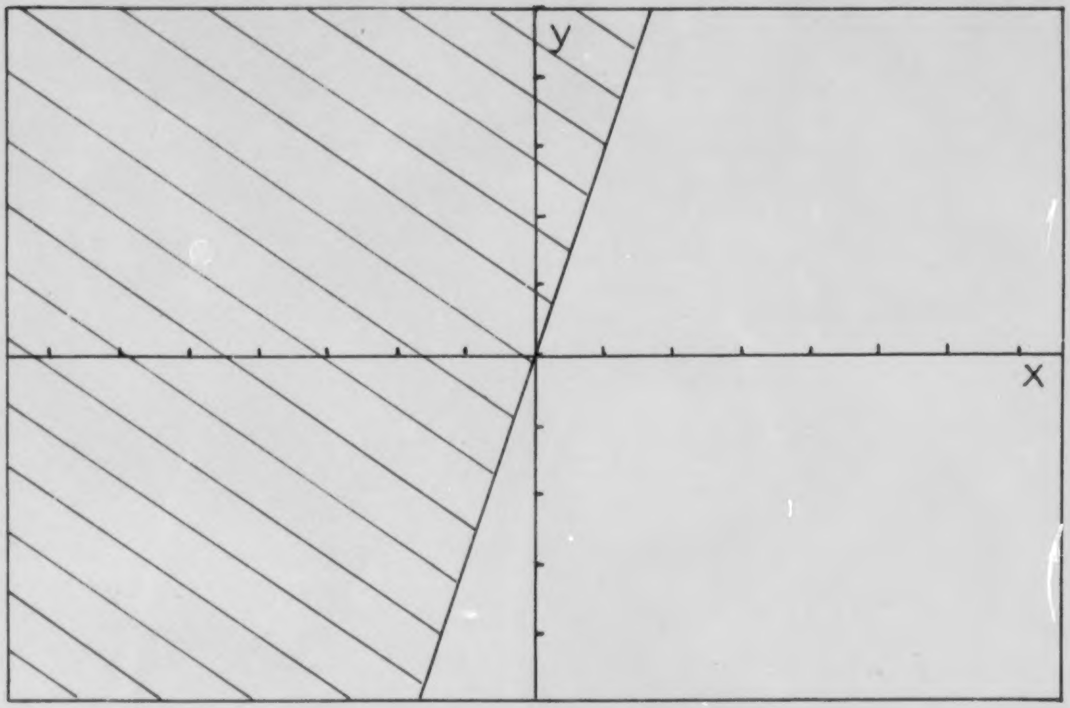


Figure 4. $y \geq 3x$

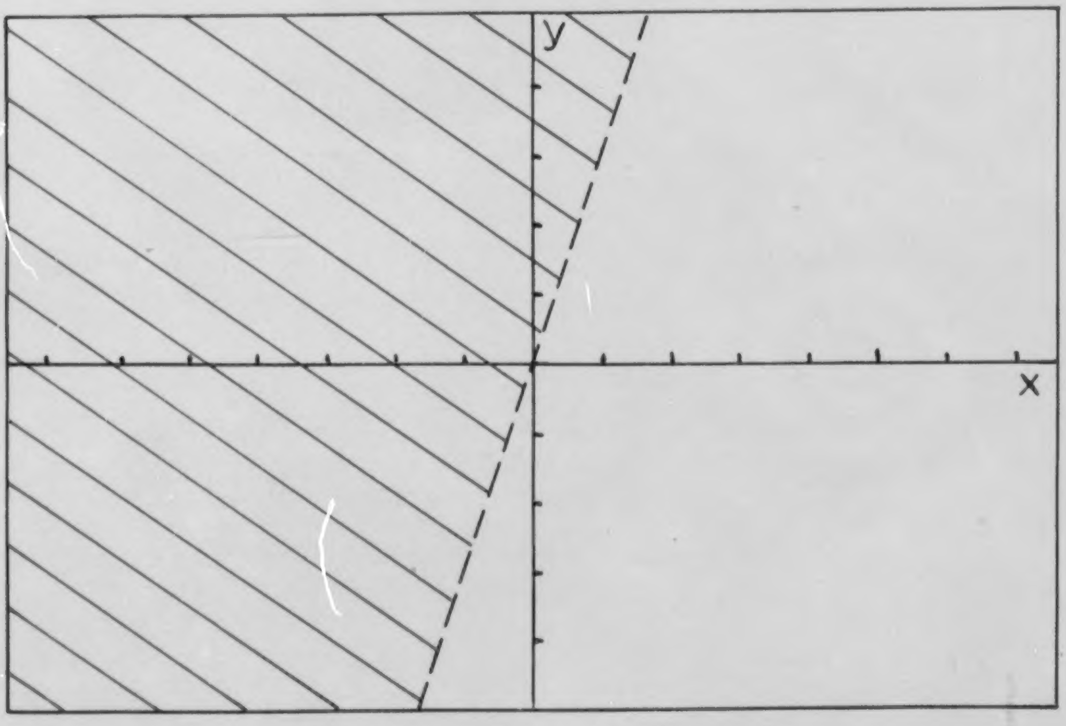


Figure 5. $y > 3x$

we make the boundary line solid, as in Figure 4, while the verb "is greater than" or "is less than" is indicated by using a dashed line for the boundary between the shaded and the unshaded regions as in Figure 5. In these two illustrations, the line is the graph of the sentence $y = 3x$. This graph, which is a line, separates the plane into two half-planes. The graph of $y > 3x$ is the half-plane such that every ordinate is greater than three times the abscissa; the boundary line, $y = 3x$, is not included in the region and is, therefore, drawn as a dashed line. The graph of $y \geq 3x$ is the half-plane including the line $y = 3x$; the line is here drawn solidly to indicate that it is included.

7. With reference to a set of coordinate axes draw the graph of the set of points associated with the ordered pairs of numbers such that each has ordinate two greater than the abscissa. What open sentence can you write for this set? Now draw the graph of the following open sentences:

   (a) $y > x + 2$; (b) $y \geq x + 2$.

   Is it possible to draw both of these graphs with reference to the same coordinate axes?

8. Given $y = |x|$. In this sentence, is y ever negative? Write the solutions for which the abscissas are: -3, -1, 0, $1\frac{1}{2}$, 2, 4. Graph the open sentence $y = |x|$ within the confines of your coordinate paper.

In problems 9 and 10:

(i) Write the sentence in the y-form.

(ii) Find at least five ordered pairs of numbers which satisfy the equation. (Why do we need no more than two points to graph the line? More points are desirable as a check.)

(iii) Draw the graph to its full extent on your paper.

9. With reference to one set of axes, draw the graphs of the following:

(a) $2x - y = 0$

(b) $3x - y = 0$

(c) $x - 2y = 0$

(d) $x + 3y = 0$

(e) $x - y = 0$

(f) $x + y = 0$

What is true about the graphs of all these open sentences?

10. With reference to one set of axes, draw the graphs of the following:

(a) $3x - 2y = 0$

(b) $3x - 2y = 6$

(c) $3x - 2y = 12$

(d) $3x - 2y = -6$

(e) $3x - 2y = -12$

What is true of the graphs of all these open sentences?

11. Draw the graphs of each of the following with reference to a different set of axes:

(a) $2x - 7y = 14$
(b) $2x - 7y > 14$
(c) $2x - 7y < 14$
(d) $2x - 7y \geq 14$

12. With reference to one set of axes draw the graphs of each of the following:

(a) $5x - 2y = 10$
(b) $2x + 5y = 10$
(c) $5x + y = 10$
(d) $3x - 4y = 6$

Which point seems to lie on three of these lines? Do its coordinates satisfy the open sentences associated with these three lines?

13. With reference to one set of axes, draw the graphs of each of the following:

(a) $2x - 3y = 10$
(b) $-x + 2y = \frac{1}{2}$
(c) $3x + 2y = 5$
(d) $\frac{1}{2}x - \frac{2}{3}y = 12$

14. Draw the graphs of the open sentences: (Find at least ten ordered pairs satisfying each equation.)

(a) $y = x^2$
(b) $y = -x^2$
(c) $y = x^2 + 1$
(d) $y = \frac{1}{x}$

Are the graphs of these open sentences lines? How do these open sentences differ from those considered in previous exercises in this chapter? Can we say that the graph of every open sentence is a straight line?

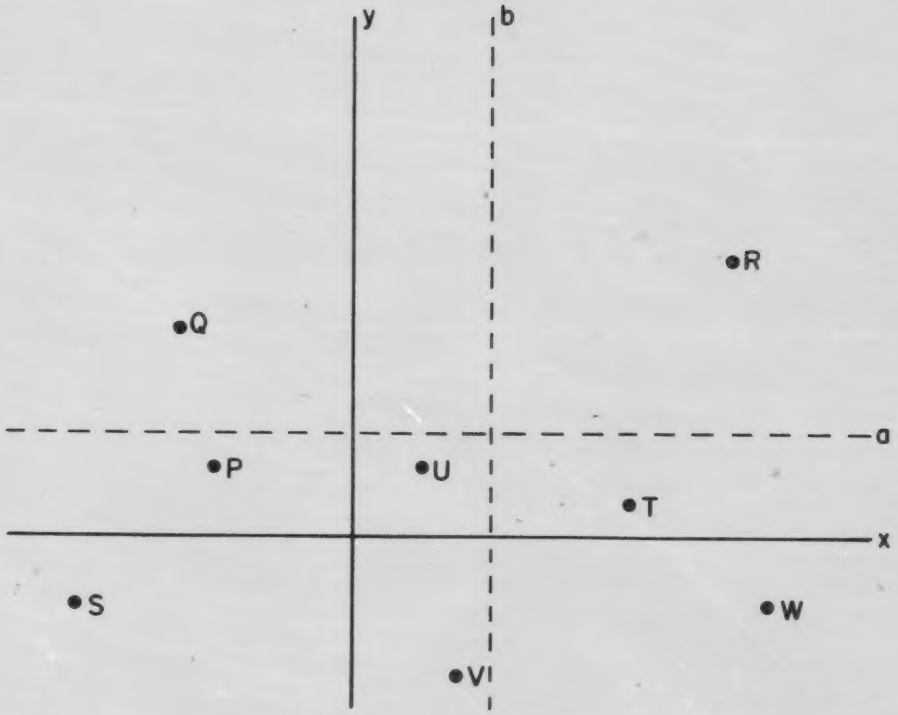


Figure 6.

15. In Figure 6 are drawn two sets of axes, as indicated: the (x, y)-axes and the (a, b)-axes.

    (a) For each of the points P through W, give the coordinates with respect to each set of axes, as is indicated below for P:

Point	(x, y)	(a, b)
P	(-4, 2)	(-8, -1)

(b) Give the (a, b) coordinates of the points whose (x, y) coordinates are (5, -5); (-3, -4); (-1, 0); (3, 5)

11 - 3. Slopes and intercepts. With reference to one set of axes, draw the graphs described in (a) through (k) below:

(a) The ordered pairs of numbers for which the ordinate is equal to the abscissa.

Fill in the blanks in the table below so that the pairs satisfy the condition:

x	-6		$-2\frac{1}{2}$		3		6
y	-6	-3		0		5.1	6

These pairs have been chosen so that the corresponding points proceed from left to right across the paper. Now connect successive points with lines. What seems to be true of these lines? Which points in the table do not lie on the line through (-6, -6) and (6, 6)? Is the point (8, 8) on the line? Extend the line in both directions, as far as possible.

What is the open sentence which describes this graph for all points in the plane? What does this line do to the angles formed by the vertical and horizontal axes?

(b) The ordered pairs of numbers for which the second is the negative of the first. Fill in the table below:

x	-6		-4.3		2.5		6.1
y		5.1		0		-4	

Could you determine pairs which fulfill the condition without making the chart? If you can do this mentally, you will not always need the chart. Can you draw a line through all of these points? Extend it as far as possible in both directions.

What is the open sentence which describes this line? How does it differ from the open sentence of the line in (a)?

(c) The ordered pair for which the ordinate is twice the abscissa. Try to draw this graph without filling a chart of pairs of numbers.

For this and the ones which follow, make a chart if necessary. Draw a complete line to get all the possible points which fulfill the condition, and write the open sentence which describes the graph.

(d) The ordered pairs for which the ordinate is six times the abscissa.

(e) The ordered pairs for which the ordinate is three times the abscissa.

(f) The ordered pairs for which the abscissa is -6 times the ordinate.

(g) The ordered pairs for which the ordinate is -3 times the abscissa.

(h) The ordered pairs for which the abscissa is one-half the ordinate.

(i) The ordered pairs for which the ordinate is minus one-half the abscissa.

(j) The ordered pairs for which the ordinate is one-sixth of the abscissa.

(k) The ordered pairs for which the ordinate is minus one-fifth of the abscissa.

Exercises 11 - 3a. Refer to the graphs just drawn to answer the following questions: (Notice that each of the sentences you have written is in the y-form.)

1. List the coefficients of x in the open sentences for which the lines lie between the graphs of "$y = x$" and "$x = 0$". What do you observe about these coefficients?
2. List the coefficients of x in the open sentences for which the lines lie between the graphs of "$y = 0$" and "$y = x$". What is true of these coefficients?
3. List the coefficients of x in the open sentences for which the lines lie between the graphs of "$y = 0$" and "$y = -x$". What is true of these coefficients?
4. List the coefficients of x in the open sentences for which the lines lie between the graphs of "$y = -x$" and "$x = 0$". What is true of these coefficients?

5. Where would you expect to find the graph of each of the following open sentences: $y = .01x$, $y = -100x$, $y = -56x$, $y = -\frac{5}{6}x$, $y = \frac{5x}{12}$, $y = \frac{24x}{25}$, $y = -\frac{25x}{24}$?

6. Make a list of information concerning a set of lines on the origin. (Note: a point lies on a line and a line is on, or lies on, a point if it passes through the point.)

7. What could you say about the graphs of equations of the form "$y = kx$", where k is a real number.

   What do you know about the graph of "$y = kx$" when k is positive? When k is negative? When k is between 0 and 1 ? When $k > 1$? When $k < -1$? When $|k| > 1$? When $|k| < 1$? When k is 0 ?

   Since k determines the direction of the line, it is called the slope of the line "$y = kx$" .

In the preceding exercises we have been considering open sentences whose graphs are lines through the origin. Now let us consider some lines which may not lie on the origin. Graph the following open sentences with reference to the same coordinate axes:

(a) $y = \frac{2}{3}x$ (b) $y = \frac{2}{3}x + 4$ (c) $y = \frac{2}{3}x - 3$

For the first of these no table of values should be necessary. We need simply note that the ordinate must be $\frac{2}{3}$ of the abscissa. In order to get points which are easy to locate we could choose multiples of 3 for values of x. To draw the graph of the second open sentence, we should note that to each ordinate in the graph of the first we add 4. How could we find the ordinates of points for the third open sentence?

What are the coordinates of the points at which lines (a), (b), and (c) intersect the vertical axis? Do you see any relation between these points and equations (a), (b), and (c) ? We call

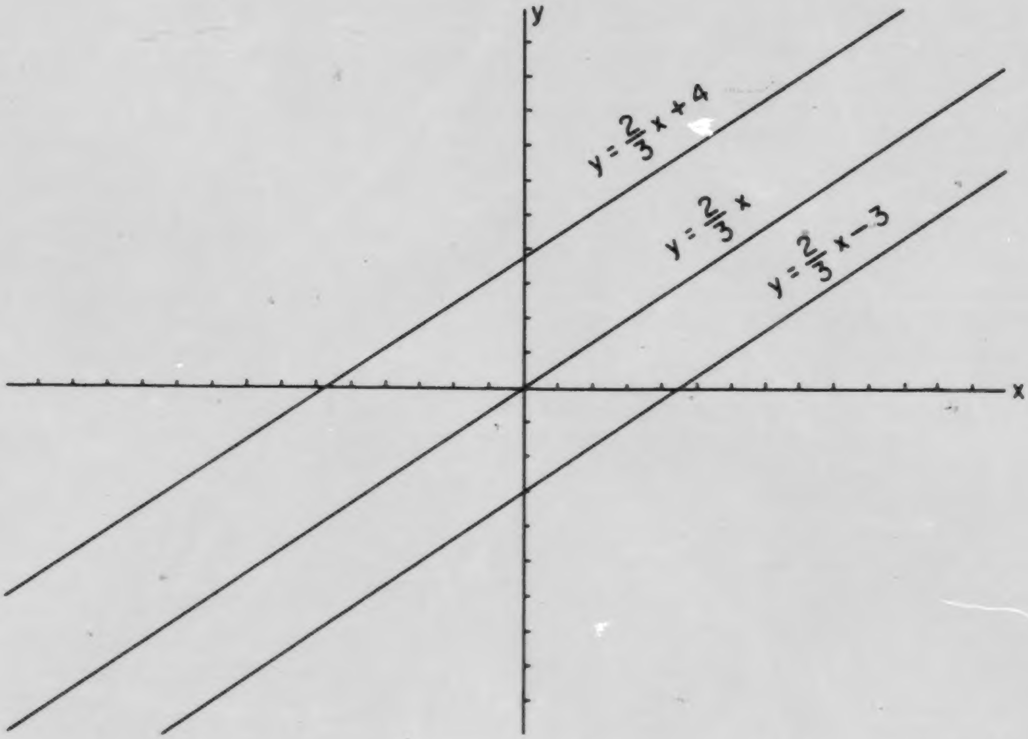


Figure 7.

0, 4, and -3 the y-intercept numbers of their respective equations. Points (0, 0), (0, 4) and (0, -3) are the y-intercepts of the respective lines. Explain how the graphs of "$y = \frac{2}{3}x + 4$" and "$y = \frac{2}{3}x - 3$" could be obtained by moving the graph of "$y = \frac{2}{3}x$".

Write two open sentences such that the absolute value of the y-intercept number is 6 and the coefficient of x is $\frac{2}{3}$. Draw the graphs of these open sentences.

In the figures which you drew at the beginning of this section, all of the lines had the same y-intercept, but many different slopes.

> The slope of a line is the coefficient of x in the corresponding sentence written in the y-form.

The slope may be either positive, negative or 0. For what positions of the line is the slope negative? 0?

> Given two points P and Q on a line, the slope of the line is the quotient of the vertical change by the horizontal change as we move from P to Q.

In Figure 8 we have a line which is the graph of "$y = \frac{5}{2}x - 3$". This line passes through points (2, 2) and (4, 7). Verify this. The ordinates are 2 and 7, respectively, and their difference, which is the vertical change, is 7 - 2, or 5. The abscissas are 2 and 4, respectively; so the horizontal change is 4 - 2, or 2. Thus the slope of the line is the quotient:

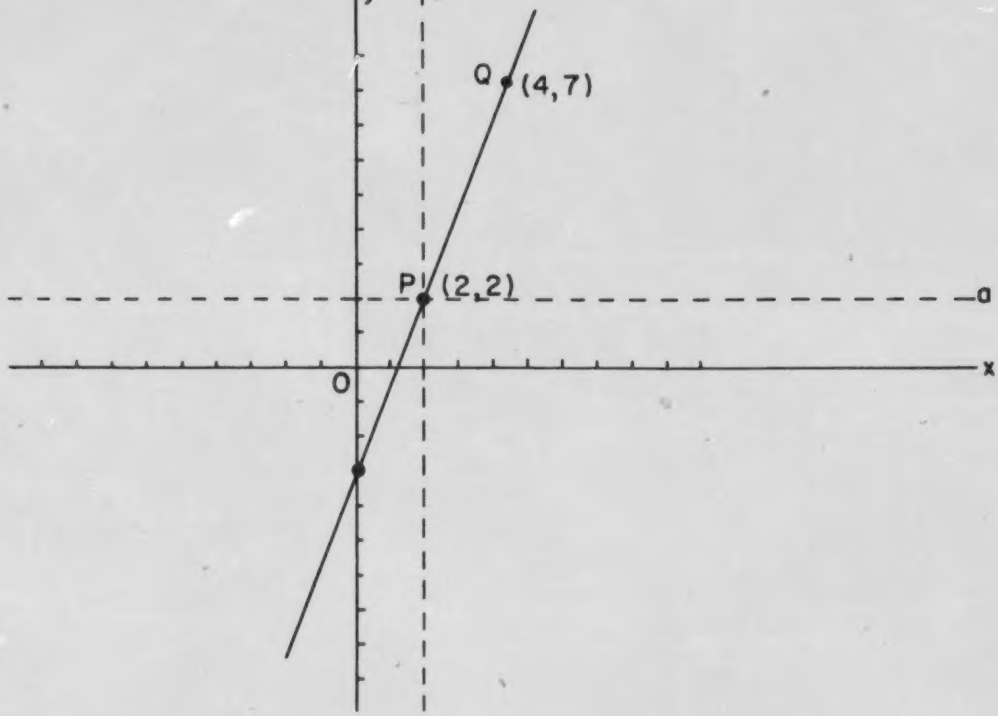


Figure 8.

$$\frac{\text{vertical change}}{\text{horizontal change}} = \frac{7 - 2}{4 - 2} = \frac{5}{2} .$$

Note the order observed in finding these differences: If the first number in the numerator is the ordinate 7 of the point (4, 7), the first number in the denominator must be the abscissa 4 of the same point. What value would we find for the slope if we used as the first number in numerator and denominator the ordinate and abscissa, respectively, of the point (2, 2) ? How would this value compare with the value just found? What is the slope of the line on the points (6, 5) and (-2, -3) ? On the points (2, 7) and (7, 3) ?

Here we have two ways of finding the slope of a line: One is to determine the coefficient of x in the y-form of its equation; the other is to compute the quotient of the change in the ordinate by the change in the abscissa from one point to another on the line. How do we know these two numbers are the same?

Consider a line such as the one drawn in Figure 9 through points P and Q. If we let P be the origin of a set of (a, b)-coordinate axes, then the line passes through the origin of the (a, b)-axes. With reference to the (a, b) axes, what are the coordinates of Q ? Now what is the equation of this line in the variables a, b ? Compare the coefficient of a in this equation with the coefficient of x in the y-form of the equation. On the other hand, the quotient of the change in ordinate by the change in abscissa from P to Q is $\frac{5}{2}$. Is this the same number as the coefficient of a in the a, b equation of the line? How could we prove these two determinations of the slope to be the same for any line?

Exercises 11 - 3b.

1. Find the slope of the line on each of the following pairs of points:

   (a) (-7, -3) and (6, 2)

   (b) (-7, 3) and (8, 3)

(c) (8, 6) and (-4, -1)
(d) (3, -12) and (-8, 10)
(e) (4, 11) and (-1, -2)
(f) (6, 5) and (6, 0)
(g) (0, 0) and (-6, -2)
(h) (0, 0) and (-7, 4)

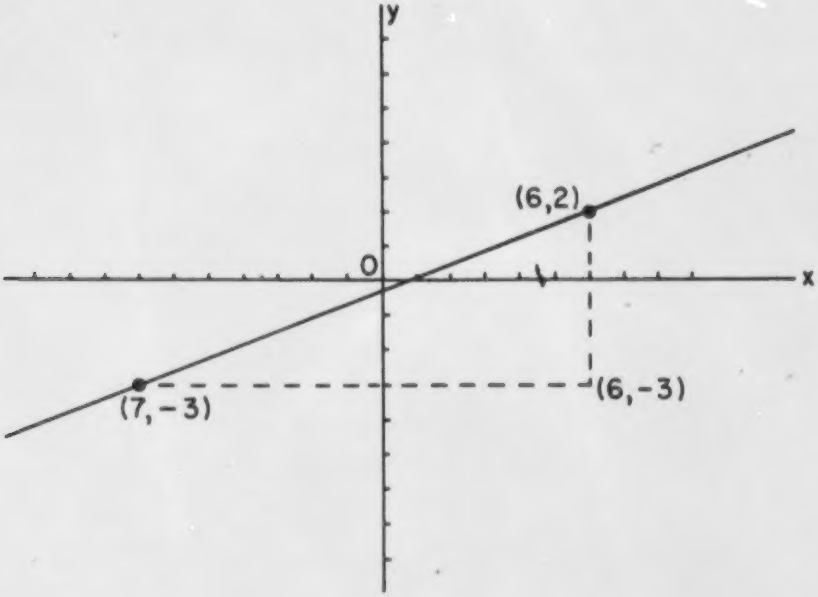


Figure 9.

In Figure 9 we note that the slope of the line is $\frac{2 - (-3)}{6 - (-7)}$ or $\frac{5}{13}$. We could check this by counting the squares, finding that from (-7, -3) to (6, 2) there are 5 units in the vertical change and 13 units in the horizontal change. It would be possible to write the open sentence of this line with a bit more information, that is, if we knew what the y-intercept number was.

In the case of the line in Figure 10 we note that it passes through points (-6, 6) and (-3, 2); so we know that its slope is $\frac{6 - 2}{(-6) - (-3)}$ or $-\frac{4}{3}$. Also, it lies on the points (-6, 6) and (0, -2),

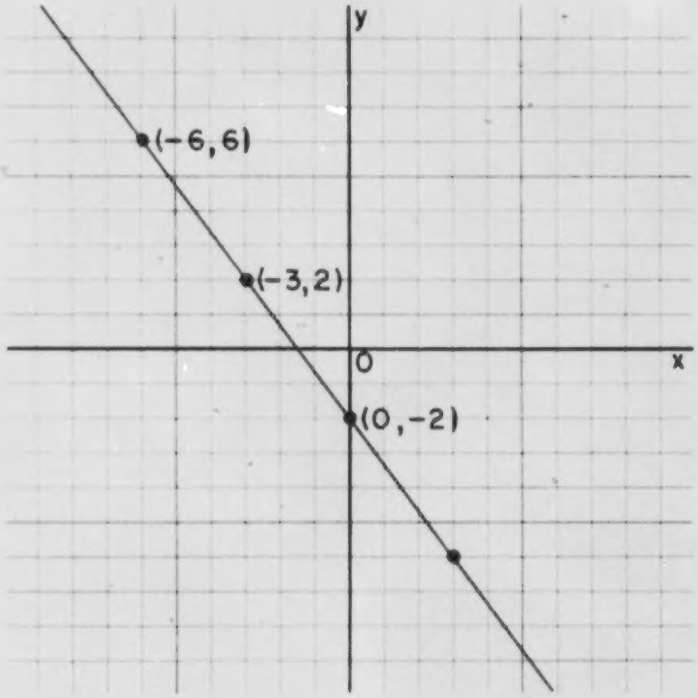


Figure 10 .

from which fact we could again determine the slope to be $\frac{6 - (-2)}{(-6) - 0} = -\frac{8}{6} = -\frac{4}{3}$, which is the same result as that found before. So we know that $y = -\frac{4}{3}x$ is the equation of a line with the same slope as the one we have in Figure 10, but which lies on the origin. We have shown that the y-intercept number of the line in Figure 10 is -2; hence, its equation is "$y = -\frac{4}{3}x - 2$" . What is the equation for a line parallel to the line in Figure 10, but which lies on the point (0, 6) ?

Exercises 11 - 3c.

1. What is the equation of a line on the point (0, 6) and parallel to the line whose equation is $y = \frac{4}{3}x - 2$?
2. What is the equation of a line parallel to $y = \frac{4}{3}x - 2$ and lying on the point (0, -12) ?
3. What is the slope of all lines parallel to $y = -\frac{4}{3}x$?
4. What is the slope of all lines parallel to $y = -\frac{2}{3}x$?
5. What is the equation of a line whose slope is $-\frac{5}{6}$ and whose y-intercept number is -3 ?
6. What is the open sentence of a line which passes through (4, 11) and (2, 4) and has y-intercept (0, -3) ?
7. What is the equation of the line which lies on (5, 6) and (-5, -4) and has y-intercept number 0 ?

Now let us see how the slope and the y-intercept can help us to draw lines. Suppose a line has slope $-\frac{2}{3}$ and y-intercept number 6. Let us draw the line as well as write its open sentence. To draw the graph, we start at the y-intercept (0, 6). Then we use the slope to locate other points on the line.

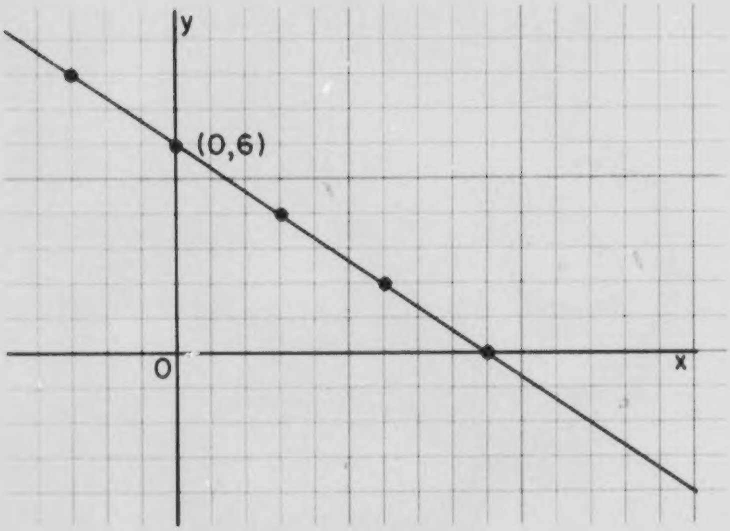


Figure 11.

The fact that the slope is $-\frac{2}{3}$ means that we shall find another point of the line three units to the left of $(0, 6)$ and two units up, and another point three units to the right and two units down. Still another is 6 units to the right and 4 units down. See Figure 11 . We can repeat this process as often as we wish, and quickly get a succession of points through which we may draw the line. Write the open sentence for the line. How would we have chosen the points with respect to $(0, 6)$ if the slope had been $\frac{2}{3}$? What would the open sentence of this line be?

Exercises 11 - 3d.

1. With reference to a set of coordinate axes, select the point $(-6, -3)$ and on this point draw the line whose slope is $\frac{5}{6}$. What is the equation of this line?

2. Draw the following lines:

   (a) on the point $(-1, 5)$ with slope $\frac{1}{2}$.

   (b) on the point $(2, 1)$ with slope $-\frac{1}{2}$.

   (c) on the point $(3, 4)$ with slope 0.

   (d) on the point $(-3, 4)$ with slope 2.

   (e) on the point $(-3, -4)$ with no slope. (What type of line has no slope? What is the difference between $\{0\}$ and $\emptyset$?

3. Consider the line on the points $(1, -1)$ and $(3, 3)$. Is the point $(-3, -9)$ on this line? (Hint: determine the slope of the line on $(1, -1)$ and $(3, 3)$; then determine the slope of the line on $(1, -1)$ and $(-3, -9)$.)

4. (a) What do the lines whose open sentences are "$y = x$", "$y = 5x$", "$y = -6x$", "$y = \frac{x}{2}$" have in common?

   (b) What do the lines whose open sentences are "$y = \frac{1}{2}x - 3$", "$y = \frac{1}{2}x + 4$", "$y = \frac{2x}{4} - 7$" have in common?

   (c) What do the lines whose open sentences are "$y = \frac{1}{2}x - 3$", "$y = \frac{3x}{4} - 3$", "$y = \frac{7x}{6} - 3$" have in common?

(d) What do the lines whose open sentences are "$x + 2y = 7$", "$\frac{1}{2}x + y = 3$", and "$2x + 4y = 12$" have in common? Show that your answer is correct by drawing the graphs of these three lines.

5. Given the equations:

(a) $3x + 4y = 12$ (b) $2x - 3y = 6$

What is the y-intercept number of each? Draw their graphs. Write each equation in the y-form. What is the slope of each line? Check with its graph.

6. Write each of the following equations in the y-form. Using the slope and the y-intercept, graph each of the lines.

(a) $2x - y = 7$ (c) $4x + 3y = 12$

(b) $3x - 4y = 12$ (d) $3x - 6y = 12$

Are you certain that the graphs of these open sentences are lines? Why?

7. Write the equation of each of the following lines:

(a) The slope is $\frac{2}{3}$ and the y-intercept number is 0 .

(b) The slope is $\frac{3}{4}$ and the y-intercept number is -2 .

(c) The slope is -2 and the y-intercept number is $\frac{4}{3}$.

(d) The slope is -7 and the y-intercept number is -5 .

(e) The slope is m and the y-intercept number is b . Can the equation of every straight line be put in this form? What about the equations of the coordinate axes?

8. Given points (0, 0) and (3, 4) with a line on them What is the slope of the line? What is its y-intercept? Write the equation of the line.

9. Write the equation of the line whose y-intercept number is 7 and which passes through the point (6, 8). What is the slope of the line? Could you write the slope as $\frac{8 - 7}{6 - 0}$?

10. What is the slope of the line on (-3, 2) and (3, -4) ? If (x, y) is a point on this same line, verify that the slope is also $\frac{y - 2}{x - (-3)}$. Also verify that $\frac{y - (-4)}{x - 3}$ is the slope. If -1 and $\frac{y - 2}{x - (-3)}$ are different names for the slope, show that the equation of the line is "$y - 2 = (-1)(x + 3)$". Show that it can also be written "$y + 4 = (-1)(x - 3)$" .

11. Write the equations of the lines through the following pairs of points:

    (a) (0, 3) and (-5, 2)
    (b) (5, 8) and (0, -4)
    (c) (0, -2) and (-3, -7)
    (d) (5, -2) and (0, 6)
    (e) (-3, 3) and (6, 0)
    (f) (-3, 3) and (-5, 3)
    (g) (-3, 3) and (-3, 5)

12. Any expression in one variable x of the form "$kx + n$" where k and n are numbers is said to be linear in x, and is called a linear expression in x, since the graph of the open sentence "$y = kx + n$" is a straight line. The

graph of "$y = kx + n$" is also called the graph of the expression "$kx + n$". Draw the graph of each of the following linear expressions:

(a) $2x - 5$ (c) $\frac{2}{3}x - 1$

(b) $-2x + \frac{1}{2}$ (d) $-\frac{3}{2}x + 2$

13. Consider a rectangle whose length is 3 units greater than its width w.

    (a) Write an expression in w whose value for each value of w is equal to the perimeter of the rectangle. Is this a linear expression in w?

    (b) Write an expression in w for the area of the rectangle. Is this a linear expression in w?

14. Consider a circle of diameter d.

    (a) Write an expression in d for the circumference of the circle. Is this expression linear in d? What happens to the circumference if the diameter is doubled? Halved? If c is the circumference, what can you say about the ratio $\frac{c}{d}$? How does the value of $\frac{c}{d}$ change when the value of d is changed?

    (b) Write an expression in d for the area of the circle. Is this expression linear in d? Is it linear in d^2? If A is the area of the circle, what can you say about the ratio $\frac{A}{d}$? What about the ratio $\frac{A}{d^2}$? Does the value of the ratio $\frac{A}{d}$ change when the value of d is changed? Does the value of $\frac{A}{d^2}$ change when d is changed?

15. In the case of the special linear expression "kx", the value of the expression is said to *vary directly as the value of the variable x*. The coefficient k is called the *constant of variation*. The value of the expression kx^2 is said to *vary directly as the square of the value of x*.

    (a) Does the circumference of a circle vary directly as the diameter? What is the constant of variation in this case? Does the area of the circle vary directly as the diameter? Does the area vary directly as the square of the diameter? What is the constant of variation?

    (b) In terms of a graph, what does the constant of variation mean?

    (c) If the constant of variation is negative, what can you say as to the way in which the value of the expression varies when you change the value of the variable?

    (d) What would be the form of an expression in one variable x such that the value of the expression varies directly as the square root of x ?

16. An automobile is moving at a constant speed along a straight road. If t is the time in hours since the start, write an expression in t whose value is the distance traveled in miles. Is this expression linear in t ? Does the distance vary directly as the time? How can you interpret the constant of variation in this example? If it is known that the automobile has traveled 25 miles at the end of 20 minutes, what is the constant of variation?

17. In the case of an expression of the form $\frac{k}{x}$, the value of the expression is said to vary inversely as the value of x. The number k is the constant of variation.

    (a) Draw the graphs of the open sentences: $y = \frac{1}{x}$; $y = -\frac{1}{x}$; $y = \frac{2}{x}$.

    (b) If the variable x is given increasing positive values, what can you say of the values of $\frac{k}{x}$? Do they increase or decrease? Does it matter whether k is positive or negative?

18. A rectangle has an area of 25 square units, and one side has length w units.

    (a) Write an expression in w for the length of the other side.

    (b) Is this a case of inverse variation? What is the constant of variation?

    (c) Draw the graph of the expression in (a).

11 - 4. Graphs of open sentences involving integers only. (Optional) In drawing graphs of open sentences, we must keep in mind that every point of a graph is associated with some pair of real numbers. Suppose we consider a sentence in which the values of the variables are restricted to integers, so that the coordinates of points on the graph must be integers. What would such a graph look like?

First let us consider the coordinate axes. Would they still be straight lines? It seems that they are sets of points such as (0, 1), (0, 2), (0, 3), etc., since we are restricting ourselves to integers, and we might wish to distinguish the axes for such cases from the coordinate axes for all real numbers. However, a series of dots would be apt to be confused with the graph itself; so we use a series of short dashes for each axis.

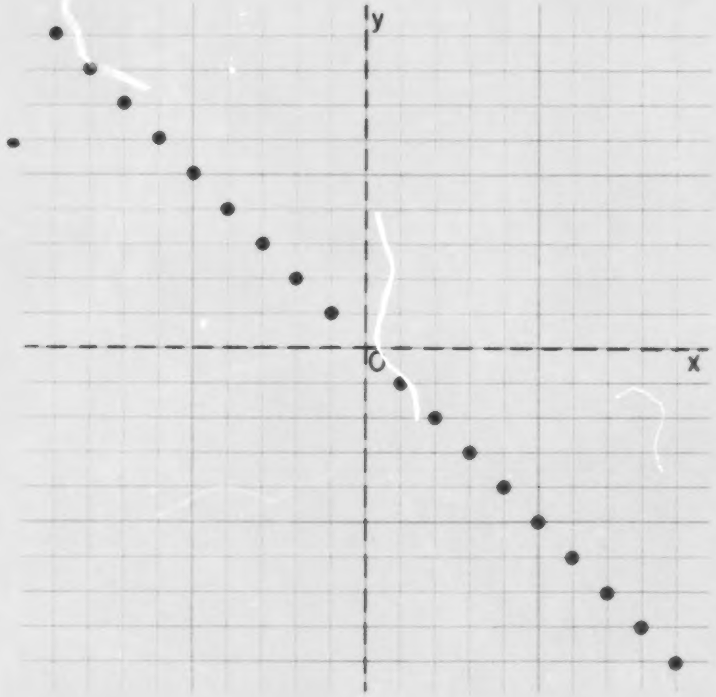


Figure 12.

What is the open sentence associated with the graph in Figure 12 ? In determining this, we first note that the graph includes points with integral coordinates only, and second that each ordinate is the opposite of the corresponding abscissa.

This may be stated as follows: "$y = -x$, where x and y are integers such that $-10 < x < 10$ and $-10 < y < 10$" .

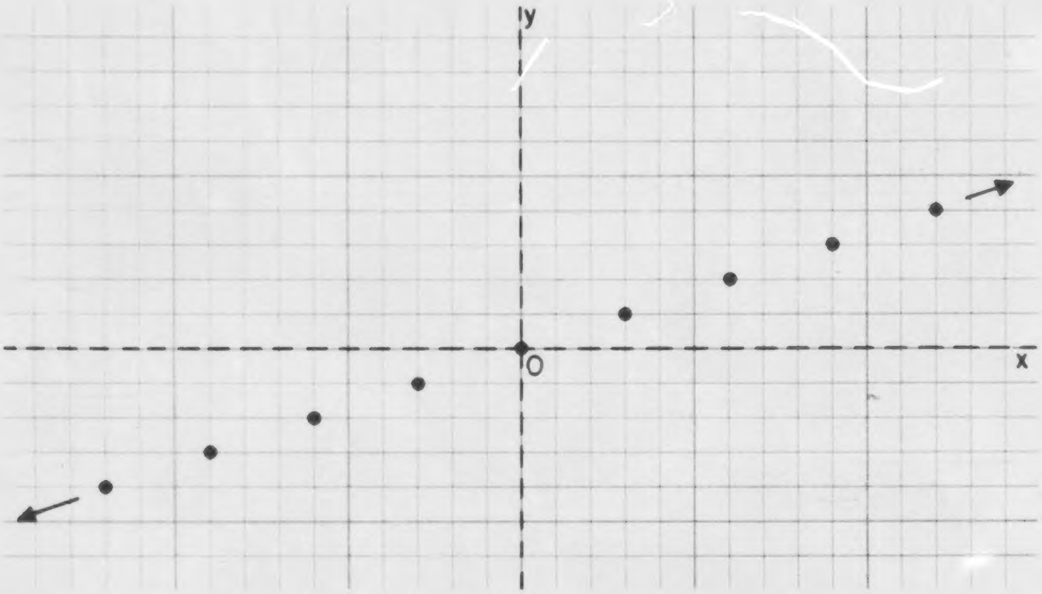


Figure 13.

In Figure 13 the arrows indicate that these points may go on beyond the limit of this diagram. Note that there are no points for $x = 1$, $x = 2$, $x = 4$, $x = -1$, and others. What do you notice about the ordinate corresponding to each abscissa, if we assume that all the points lie on a straight line, as these points seem to indicate? We would write the open sentence: "$y = \frac{x}{3}$ where x and y are integers" . Why can the abscissa not be 1 or 2 ?

Consider Figure 14. For this set of twelve points it seems there is no simple open sentence. Can you describe the limitations on the abscissas? What statement can you make about the ordinates?

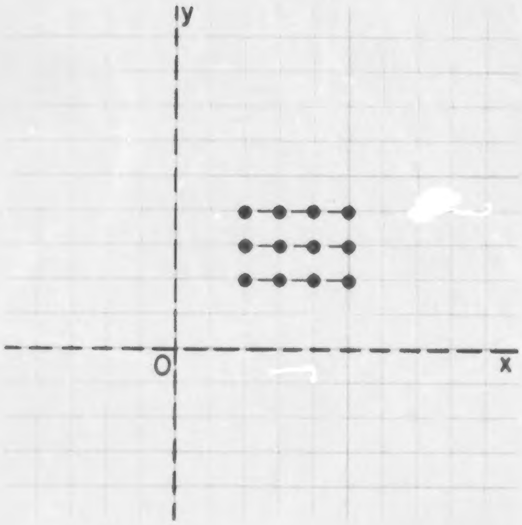


Figure 14.

These facts could be stated in a compound open sentence as follows: "$1 < x < 6$ and $1 < y < 5$, where x and y are integers" .

Notice that here the connective for the compound sentence is and; note also that the points whose coordinates make the sentence true are only those which belong to the truth sets of both parts of the compound sentence.

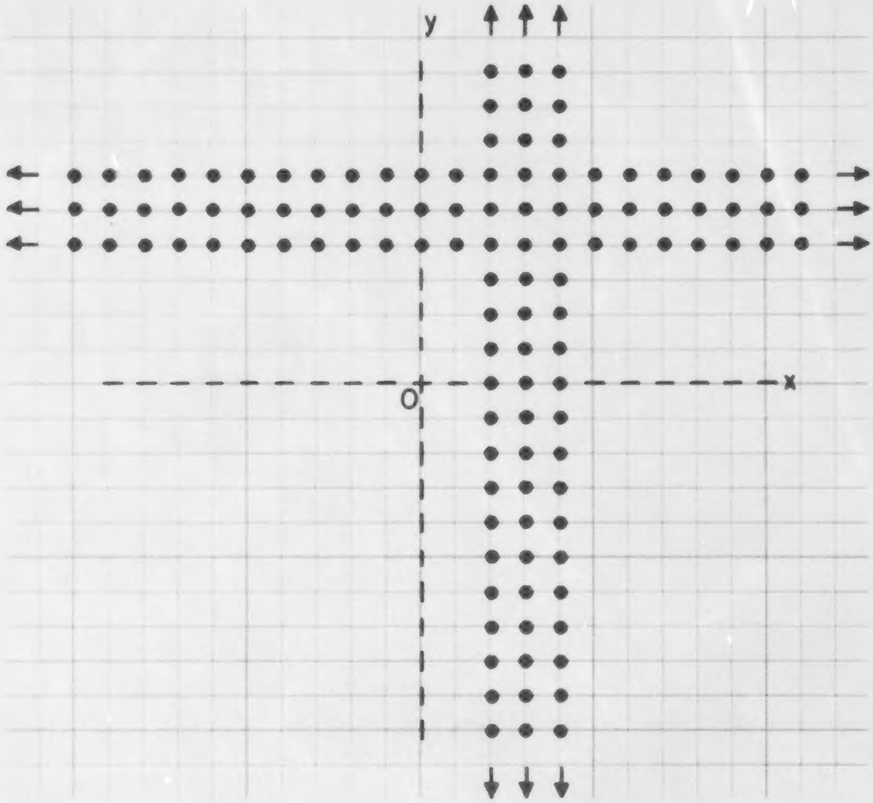


Figure 15.

In Figure 15 a different situation exists. Let us see what open sentence will describe this graph. The three horizontal rows of dots could be the graph of the sentence: "$3 < y < 7$, where x and y are integers" . Then we write a sentence which describes the three vertical rows of dots: "$1 < x < 5$ where x and y are integers". The open sentence which describes the total set of points is "$1 < x < 5$ or $3 < y < 7$, where x and y are integers". Another way of stating this would be: "$2 \leq x \leq 4$ or $4 \leq y \leq 6$ where x and y are integers". Notice that the connective here is or, and that the graph includes all points which belong to the truth sets of either of the two parts of the compound sentences, or to both of them.

Exercises 11 - 4.

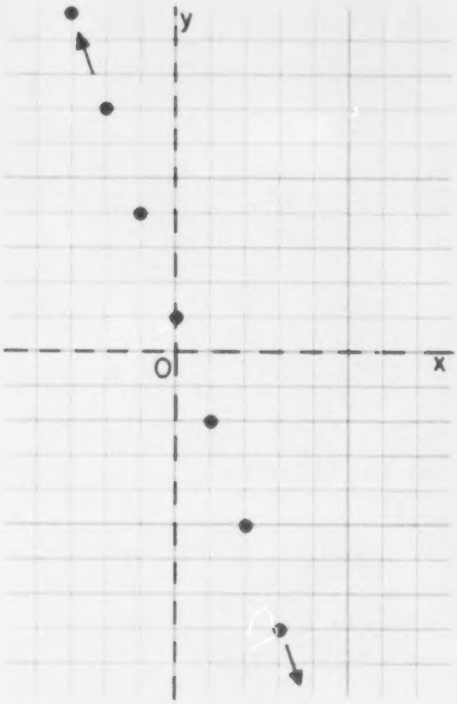
1. With reference to separate sets of coordinate axes, and for x and y integers, draw the graph of each of the following:
   (a) $y = \frac{x}{2}$, for $-6 < x < 6$
   (b) $y = 3x - 2$
   (c) $y = 2x + 4$

2. Draw the graphs of each of the following with reference to a separate set of coordinate axes:
   (a) $-3 < x < 2$ and $-2 < y < 1$, where x and y are integers.
   (b) $-3 < x < 2$ or $-2 < y < 1$, where x and y are integers.
   (c) $5 \leq x \leq 6$ or $1 \leq y \leq 3$, where x and y are integers.
   (d) $5 \leq x \leq 6$ and $y = 0$, where x and y are integers.

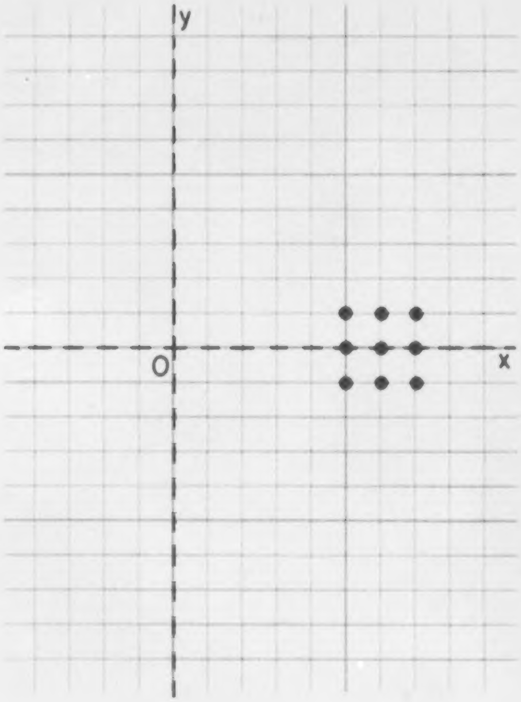
3. Write a compound open sentence whose truth set is $\{(-1, 3)\}$.

4. Write open sentences whose truth sets are the following sets of points:

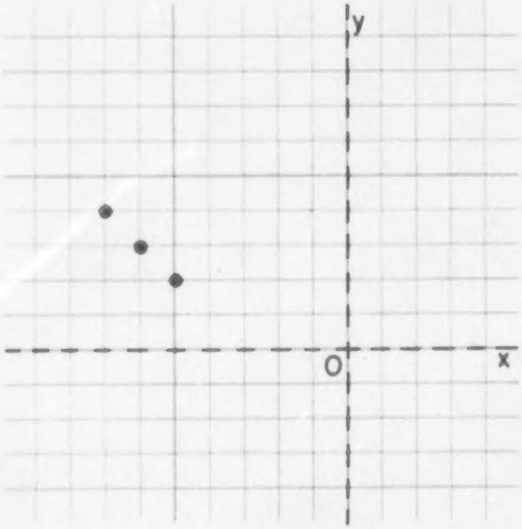
(a)



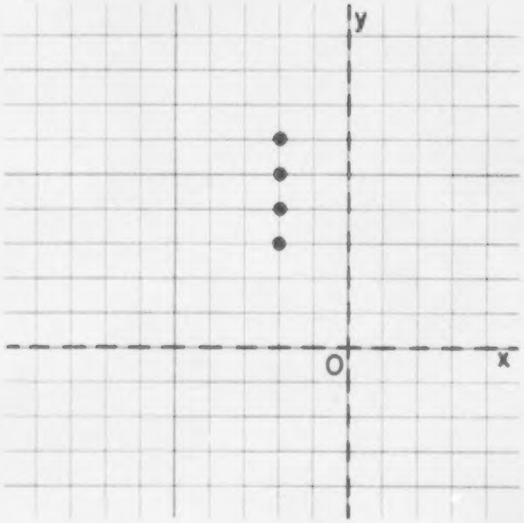
(b)



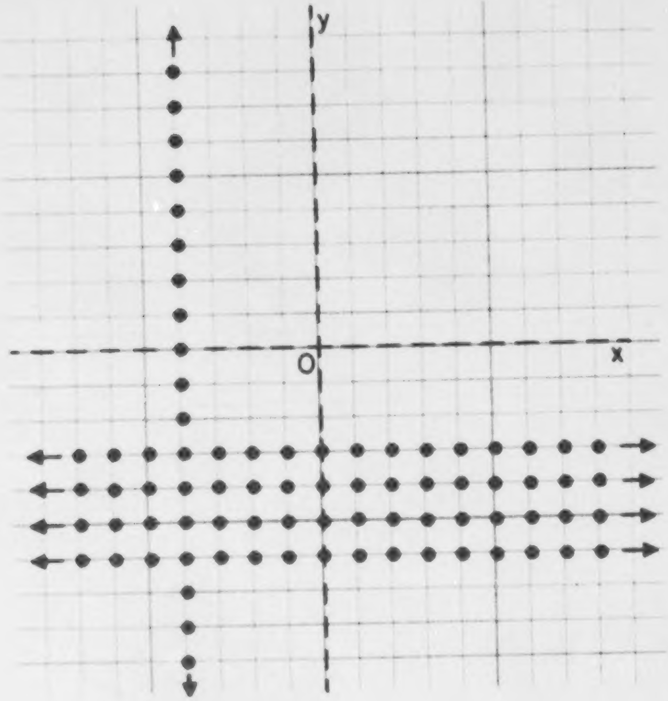
(c)



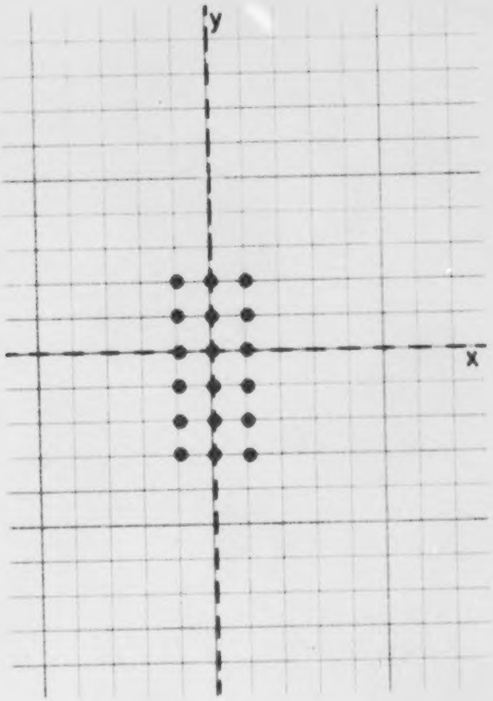
(d)



(e)



(f)



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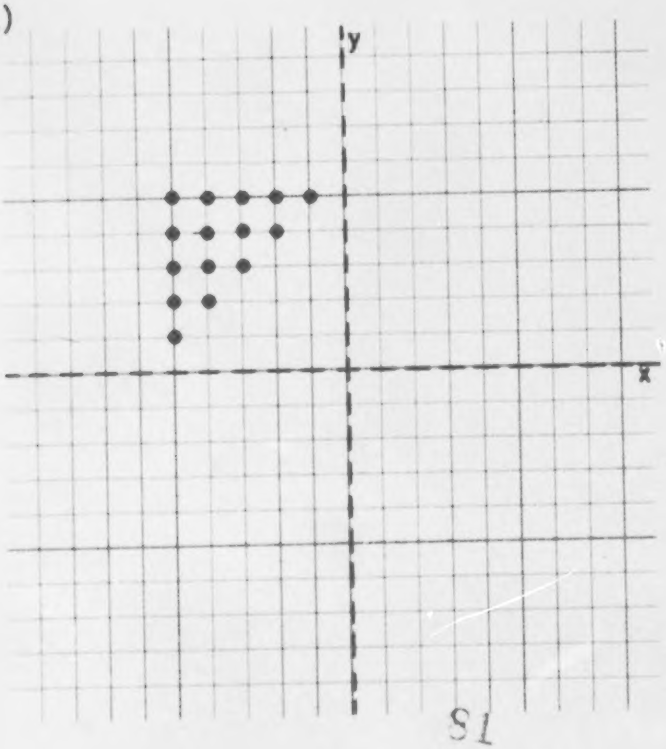


5. With reference to separate sets of coordinate axes, draw the graphs of the following:

   (a) $x + 5y = 15$, where x and y are integers.

   (b) $x + 5y = 15$, where x and y are real numbers.

   How do the graphs of (a) and (b) differ? Name some points on one graph which are not on the other.

   (c) $x + 5y = 15$, where x and y are rational numbers.

   How does this differ from the other graphs? If (x, y) is a point on the line $x + 5y = 15$ and if x is rational, what about y ?

11 - 5. Graphs of open sentences involving absolute value. Consider the equation "$|x| = 3$". This sentence is equivalent to the sentence "$x = 3$ or $x = -3$". What would its graph look like? First consider the graph for "$x > 0$ and $|x| = 3$", namely, "$x = 3$". The graph of this open sentence is a straight line parallel to the vertical axis and three units to the right of it. But if $x < 0$ and $|x| = 3$, then $x = -3$,

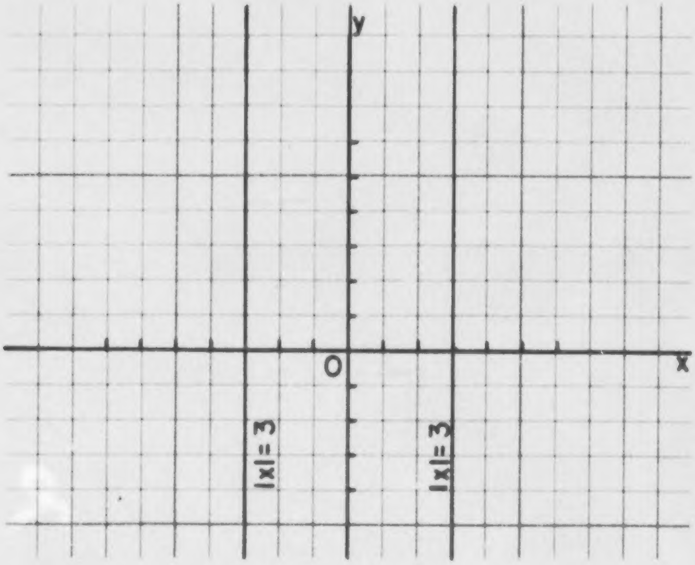


Figure 16.

and we have also a second line parallel to the vertical axis and three units to the left of it. Hence, the complete graph of $|x| = 3$ consists of two lines which are the graphs of "$x = 3$", "$x = -3$", as in Figure 16. Describe and draw the graphs of:

(a) $|x| = 5$ |b) $|x| = 7$ (c) $|y| = 2$ (d) $|y| = 3$

For what value of k will the graph of $|x| = k$ be a single line?

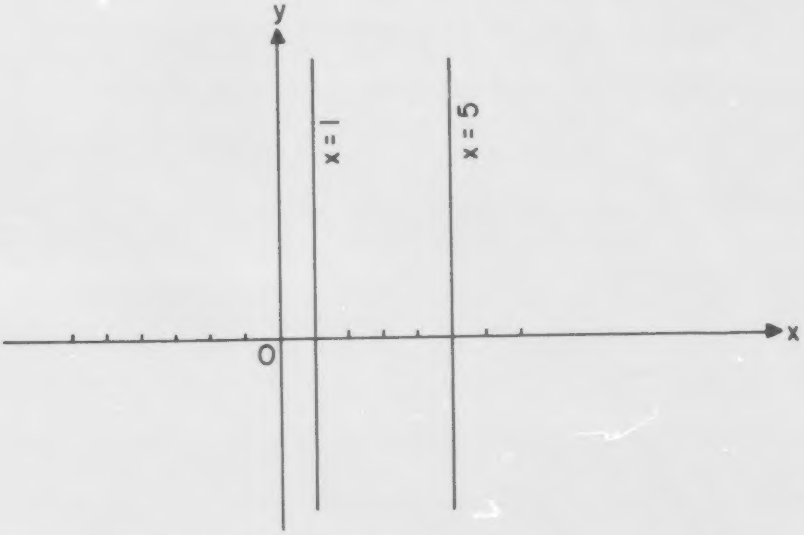


Figure 17.

Consider the graph of the open sentence "$|x - 3| = 2$".

This sentence is equivalent to "$x - 3 = 2$ or $x - 3 = -2$". What would its graph look like? First consider the open sentence is a straight line parallel to the vertical axis and five units to the right of it. The graph of "$x - 3 = -2$", that is, of "$x = 1$", is a line parallel to the vertical axis and 1 unit to the right of it. Hence the complete graph of "$|x - 3| = 2$" consists of two lines, one the graph of "$x = 5$" and the other the graph of "$x = 1$", as in Figure 17. Check on the graph that one of these lines is two units to the right of $x = 3$, and the other is two units to the left of $x = 3$.

Exercises 11 - 5a.

1. Draw the graph of each of the following open sentences with reference to a different set of axes:

   (a) $|x - 4| = 2$
   (b) $|y - 2| = 3$
   (c) $|y| = 5$
   (d) $|x + 1| = 2$
   (e) $|x + 3| = 1$
   (f) $|y + 2| = 3$

2. Draw the graph of each of the following open sentences with reference to a different set of axes:

   (a) $|x| > 2$
   (b) $|x| \geq 2$
   (c) $|x| < 5$

3. Draw the graph of each of the following with reference to a different set of axes:

   (a) $y > 2x + 4$
   (b) $y < \frac{2x}{3} + 7$
   (c) $y \geq \frac{3x}{4} - 1$
   (d) $y \leq 2x - 1$

4. Draw the graph of each of the following with reference to a different set of axes:

   (a) $2x + y > 3$
   (b) $x + 2y \geq 4$
   (c) $x - 2y \leq 4$
   (d) $2x - y \leq 3$

5. Write the open sentences for which lines (1), (2), (3), and (4) in the figure are the graphs. Notice that (4) actually is a pair of lines.

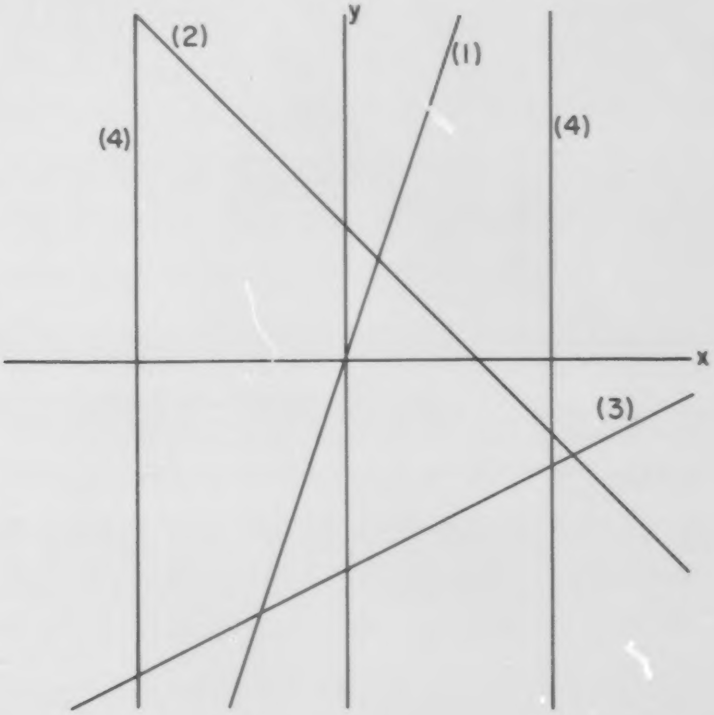


Figure for Problem 5.

6. Draw a set of coordinate axes. With reference to these axes, locate three points from each of the sets described below. For each set draw a line through the three points.

   (a) The second coordinate of the ordered pair is twice the first.

   (b) The second coordinate of the ordered pair is 5 more than one-half the first.

   (c) The second coordinate of the ordered pair is one-half the first.

(d) The second coordinate of the ordered pair is the opposite of the first.

Let us consider the open sentence "$y = |x|$". Whether x is positive or negative, what is true of the absolute value of x? What, then, must be true of y for every value of x except 0 ? What is the value of y for $x = 0$?

x	-3	-2	-1	0	1	2	3
$\|x\|$	3	2	1	0	1	2	3

From Figure 18 we notice something new to us: the graph of the simple sentence "$y = |x|$" turns out to be the two sides of a right angle. Can you guess why this is a right angle? Is it possible to have a simple equation whose graph would be two lines which do not form a right angle? Suggest one.

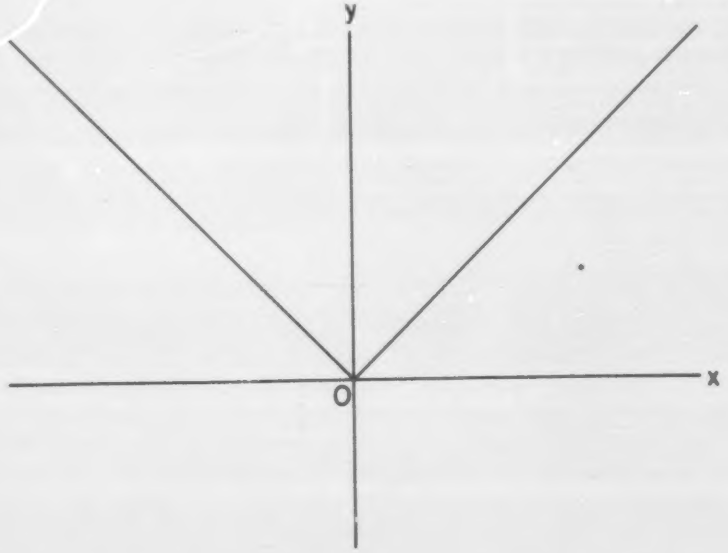


Figure 18.

Exercises 11 - 5b.

1. Draw the graph of each of the following with reference to a separate set of axes:

   (a) $y = 2|x|$
   (b) $y = \frac{1}{2}|x|$
   (c) $y = -|x|$
   (d) $y = -2|x|$
   (e) $x = -|y|$
   (f) $x = |-2y|$

2. Draw the graph of each of the following with reference to a separate set of axes:

   (a) $y = |x| + 3$
   (b) $y = |x| - 7$
   (c) $y = 2|x| + 1$
   (d) $x = |y| + 3$
   (e) $x = 2|y| - 1$
   (f) $y = -|x| - 1$

3. Draw the graph of each of the following with reference to a separate set of axes:

   (a) $y = |x - 2|$
   (b) $y = |x + 3|$
   (c) $y = 2|x + 3|$
   (d) $y = |x + 3| - 5$
   (e) $y = \frac{1}{2}|x - 1| + 3$

4. How would you get each of the graphs in Problems 1 (c), (e), 2 (a), (b), (d), (f), 3(a), (b), (d) from the graph of either $y = |x|$ or $x = |y|$ by rotating or sliding the graph? Examples: The graph of "$y = |x - 2|$" can be obtained by sliding the graph of "$y = |x|$" to the right 2 units. The graph of "$x = -|y|$" can be obtained by rotating the graph of "$x = |y|$" about the y-axis. The graph of "$y = |x| - 7$" can

be obtained by sliding the graph of "$y = |x|$" down 7 units.

5. What does the graph of $|x| + |y| = 5$ look like? Let us make a chart first. Suppose we start with the intercepts. Let $y = 0$ and get all possible values of x which will make the sentence true. Then let $x = 0$, and get the values of y. Now fill in some of the other possible values. Suppose $x = 6$, what can you say about possible values for y? If $x = 3$, then $|x| = 3$, and $|y| = 2$; what possible values may y have? Fill in the blanks in the table below, and then draw the graph. How would you describe the figure?

x	-5	-3	-3	-1	-1	0	0	1	1	3	3	5
$\|x\|$	5	3	3	1		0	0					5
$\|y\|$	0		2	4		5	5					0
y	0		-2	4		5	-5					0

We could write four open sentences from which we could get the same graph, provided we limited the values of x:

$$x + y = 5, \text{ and } 0 \leq x \leq 5,$$
$$x - y = 5, \text{ and } 0 \leq x \leq 5,$$
$$-x + y = 5, \text{ and } -5 \leq x \leq 0,$$
$$-x - y = 5, \text{ and } -5 \leq x \leq 0.$$

With reference to one set of axes, draw the graphs of the four open sentences stated above. Why was it necessary to limit the values of x?

6. Draw the graph of each of the following with reference to a separate set of axes:

(a) $|x| + |y| > 5$ (c) $|x| + |y| \leq 5$

(b) $|x| + |y| < 5$ (d) $|x| + |y| \not< 5$

7. Make a chart of some values which make the open sentence

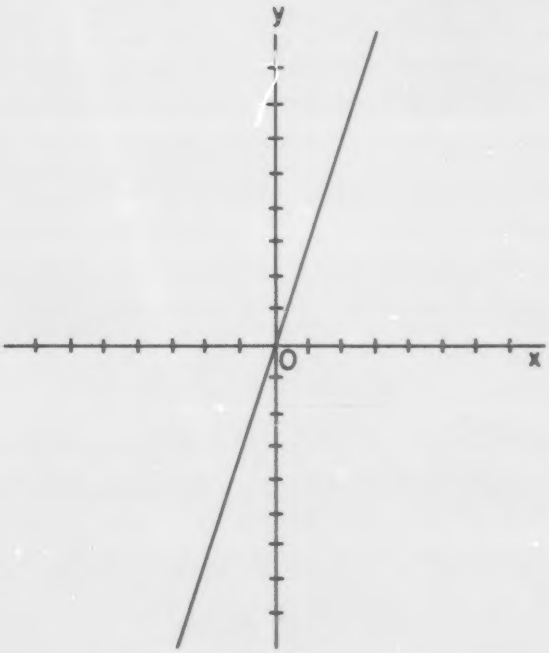
$$|x| - |y| = 3$$

true, and draw the graph of the open sentence. Write four open sentences, as in problem 3, whose graphs form the same figure.

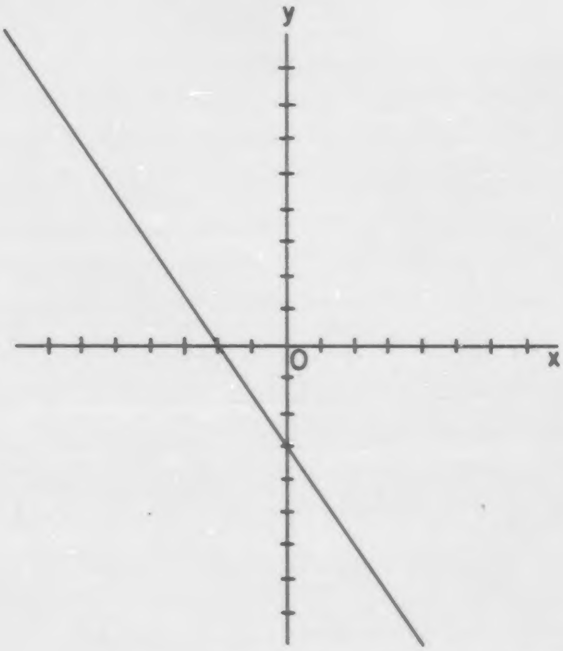
11 - 6. Review Exercises.

1. For each of the following graphs, write its open sentence, assuming in each case that one square measures one unit:

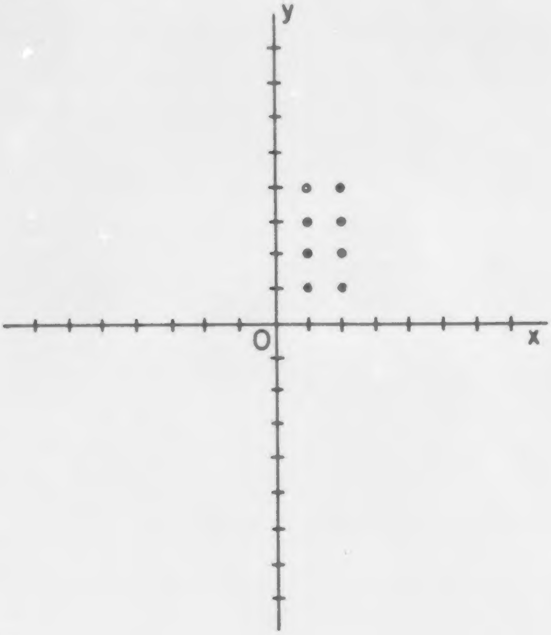
(a)



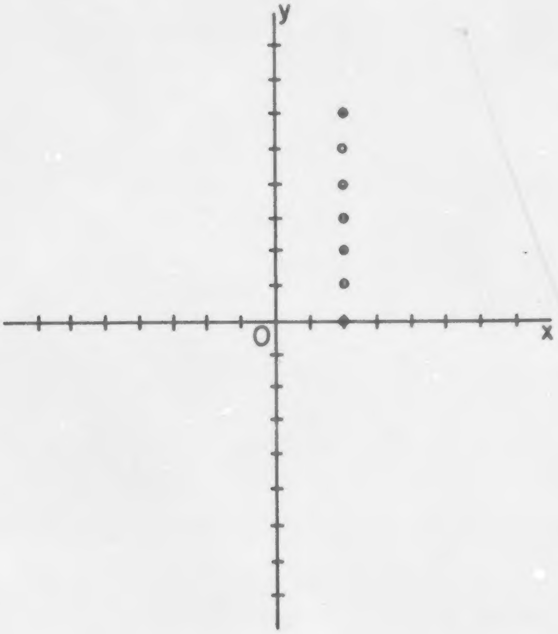
(b)



(c) (Optional)



(d) (Optional)



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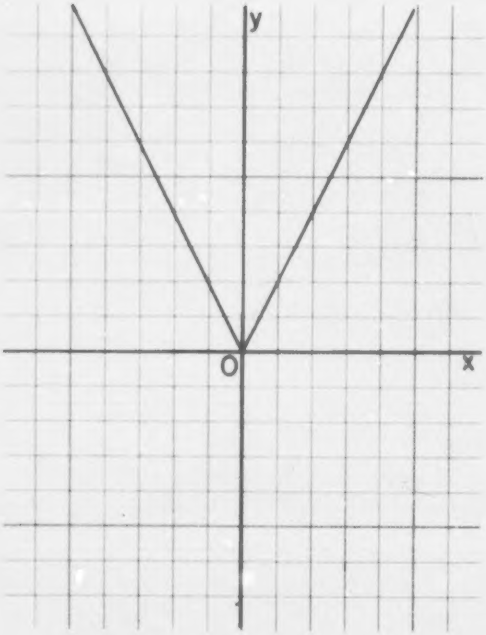
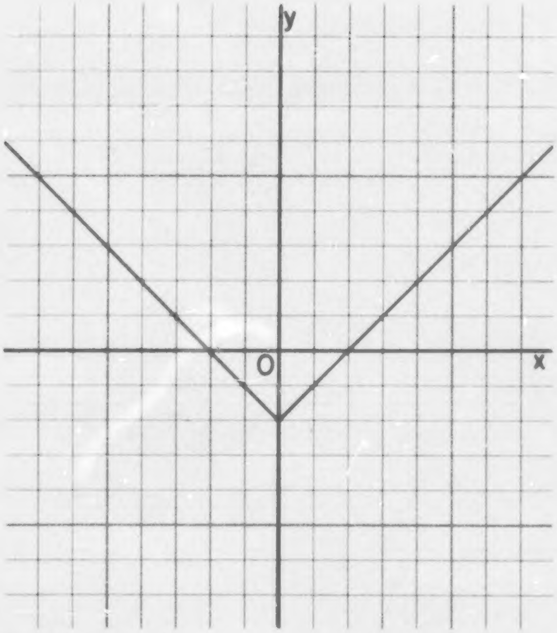


(f)



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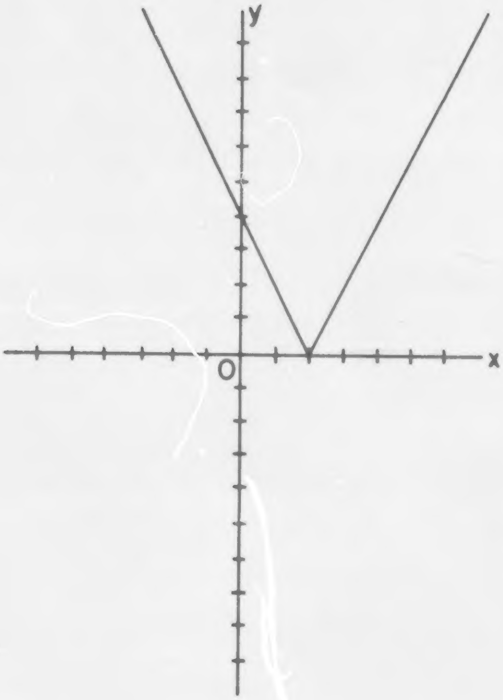
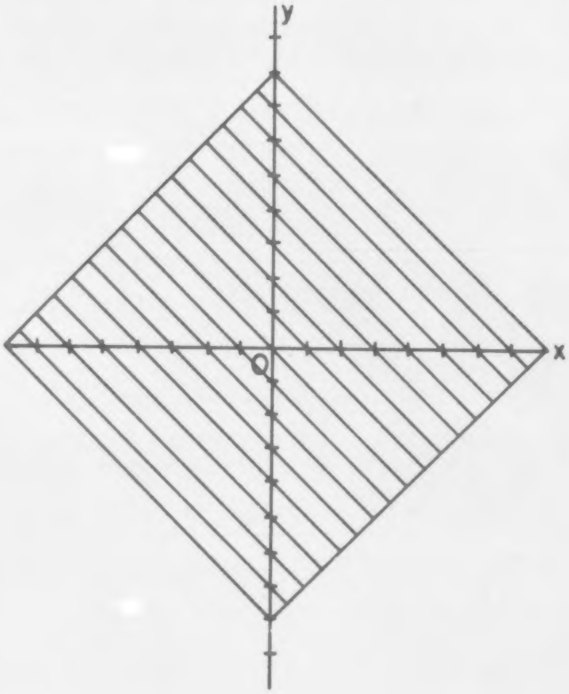
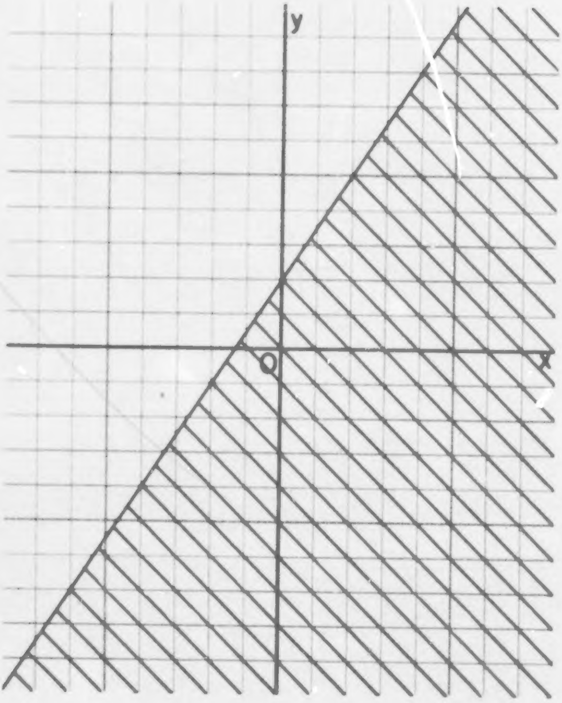


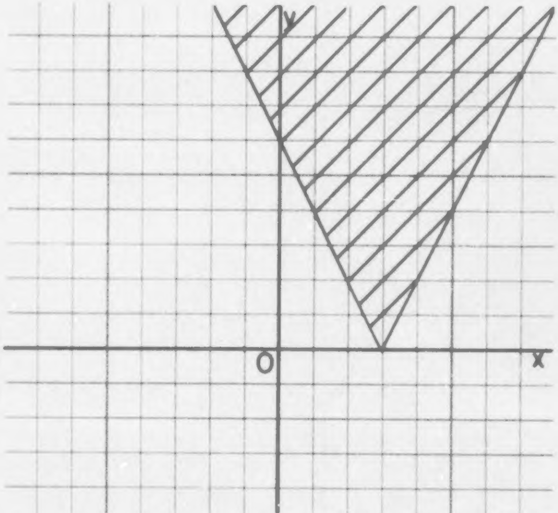
(h)



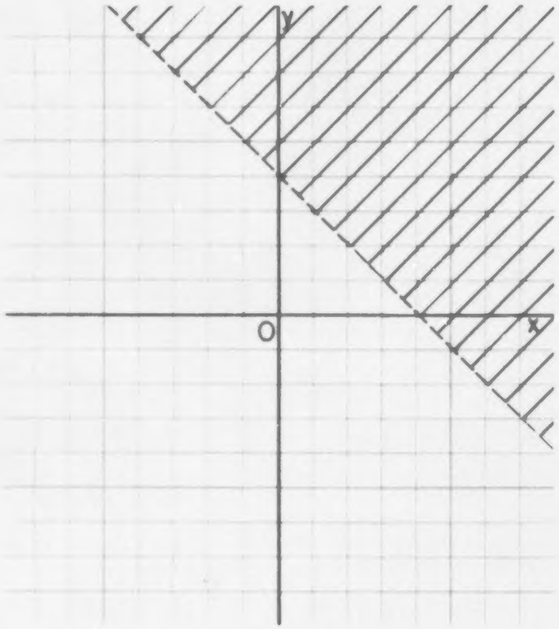
(i)



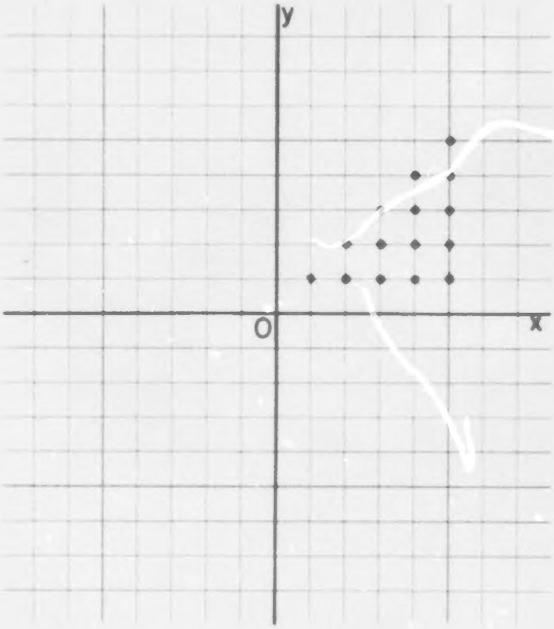
(j)



(k)



(l) (Optional)



2. Draw the graph of each of the following open sentences:

(a) $y \leq 3x$

(b) $y = \frac{x}{2} + 7$

(c) $y < \frac{x}{2} - 5$

(d) $y > 3$

(e) $x < 1.5$

(f) $x + y = 0$

(g) $|x| + |y| = -2$

(h) $3y \geq 2x - 1$

(i) $x = 3$ and $y = -1$

(j) $x + y \leq -2$

(k) $3y - 12 = 0$

3. Draw a set of coordinate axes, designating them as the (x, y)-axes. Through point (2, -1) draw a pair of (a, b)-axes, making the a-axis parallel to the x-axis and the b-axis parallel to the y-axis. Locate the following points with

reference to the (x, y)-axes: A(3, -5), B(-5, 3), C(-2, -5), D(0, 3), E(0, -3), G(-5, -1), H(-4, 3), I(6, 0), J(-6, 0), K(2, 6). Make a table giving the coordinates of each of these points with reference to the (a, b)-axes.

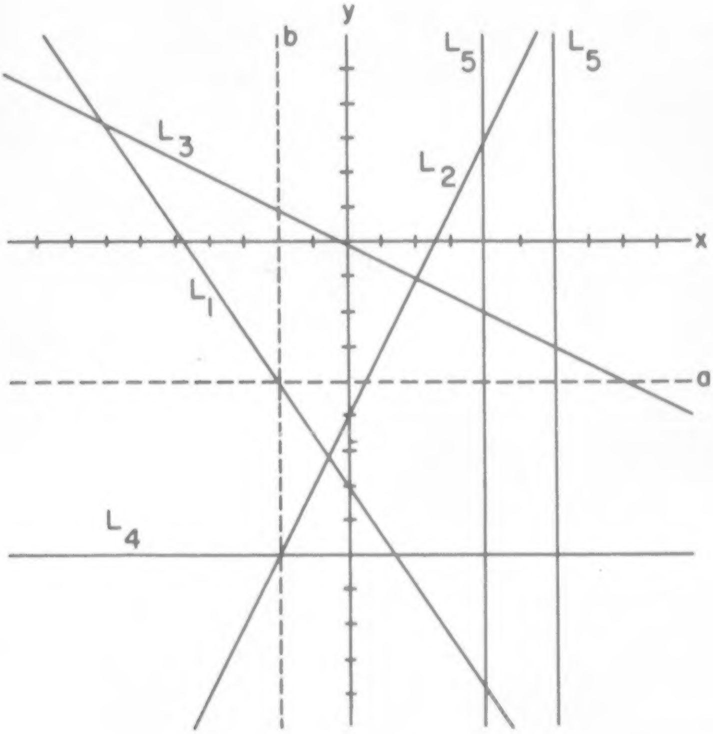


Figure for Problem 4.

4. Give two equations for each of the lines in Figure 21, one with reference to the (x, y)-axes, the other with reference to the (a, b)-axes. (Note that L_5 is a pair of lines.)

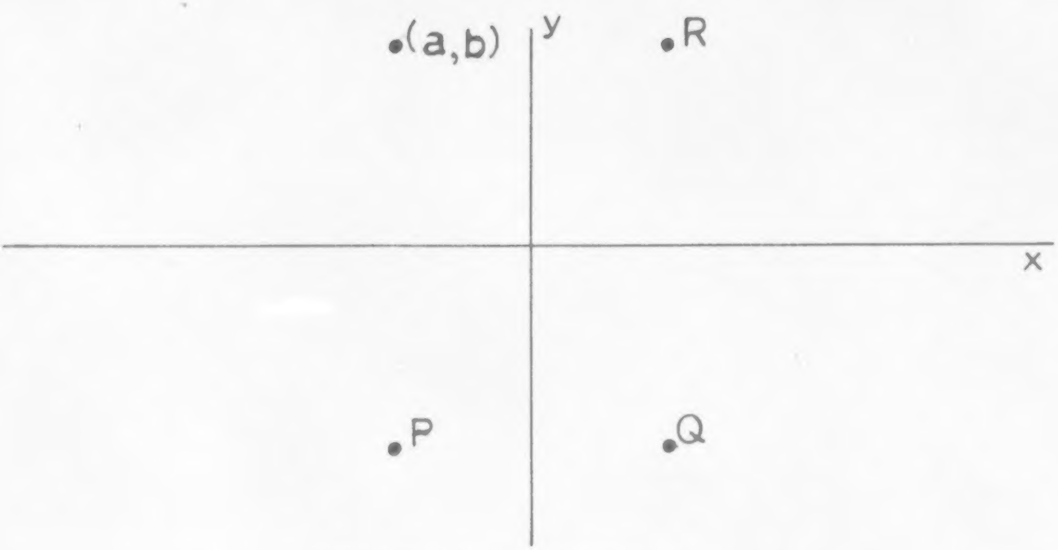


Figure for Problem 5.

5. If the point (a, b) is in the second quadrant

   (a) is a positive?

   (b) is b positive?

   (c) If the coordinates of P, Q, and R have the same absolute values as the abscissa and ordinate of (a, b), state the coordinates of P, Q, and R in terms of a and b.

6. If (c, d) is a point in the third quadrant, in which quadrant is the point (c, -d)? The point (-c, d)? The point (-c, -d)?

7. Draw the graph of "$y = 3x + 4$". What happens to this graph when its equation is changed as follows?

   (a) $y = 3(-x) + 4$

   (b) $y = -(3x + 4)$

   (c) $y = (3x + 4) - 3$

   (d) $y = 3(x - 2) + 4$

8. Draw the graph of "$y = 2|x|$". Give the equation of the graph which results from each of the following changes:
   - (a) The graph is rotated one-half revolution about the x-axis.
   - (b) The graph is moved 3 units to the right.
   - (c) The graph is moved 2 units to the left.
   - (d) The graph is moved 5 units up.
   - (e) The graph is moved 2 units to the right and 4 units down.

9. (a) With reference to one set of axes, draw the graphs of:

$$2x + y - 5 = 0$$

$$6x + 3y - 15 = 0$$

What is true about these two graphs? Now look at the equations; how could you get the second equation from the first?

(b) What is true of the graphs of:

$$Ax + By + C = 0$$

and

$$kAx + kBy + kC = 0 \quad \text{for any non-zero } k?$$

(c) Under what condition will the graphs of

$$Ax + By + C = 0$$

and

$$Dx + Ey + F = 0$$

be the same line? If the graphs are the same line, what is true of the ratios $\frac{A}{D}$, $\frac{B}{E}$, and $\frac{C}{F}$?

10. (a) With reference to one set of axes, draw the graphs of

$$3x - 4y - 12 = 0,$$
$$12x - 16y - 32 = 0.$$

What is true about these two graphs? What is true about the coefficients of x and y in these equations?

(b) What is true of the graphs of

$$Ax + By + C = 0$$

and

$$kAx + kBy + D = 0 \quad \text{for any non-zero } k?$$

(c) Under what conditions will the graphs of

$$Ax + By + C = 0$$
$$Dx + Ey + F = 0$$

be parallel lines? If the graphs are parallel lines, what equal ratios are there among the coefficients?

11. Draw the graphs of the equations

(a) $7u + 3t - 4 = 0$

(b) $2s - 5v - 1 = 0$

Do the graphs depend on the choices you made for the first variable? Explain why we speak of sentences with two ordered variables.

12. Suppose a point with coordinates (a, b) is moved into the point $(a, -b)$. Describe this in terms of opposites.

Describe it in terms of a rotation. Answer the following questions, and locate the points referred to in parts (a) and (b).

(a) What points do the following points go into:
$(2, 1)$, $(2, -1)$, $(-\frac{1}{2}, 2)$, $(-2, -3)$, $(3, 0)$,
$(-5, 0)$, $(0, 5)$, $(0, -5)$.

(b) What points go into the points listed in (a) above?

(c) What point does $(a, -b)$ go into?

(d) What point does $(-a, b)$ go into?

(e) What point goes into (a, b)?

(f) What points go into themselves?

13. Suppose a point with coordinates (a, b) is moved into the point $(a - 3, b + 2)$. How can you obtain this by moving all the points of the plane? Answer the following questions and locate the points:

(a) What points do the following points go into:
$(1, 1)$, $(-1, -1)$, $(-2, 2)$, $(0, -3)$, $(3, 0)$?

(b) What points go into the above points?

(c) What point does $(a, b - 2)$ go into?

(d) What point goes into $(-a, -b)$?

(e) Which points go into themselves?

(f) Describe how the points are moved if (a, b) is moved into $(a, b - 2)$.

Chapter 12

Systems of Equations and Inequalities

12 - 1. Systems of equations. We began a study of compound sentences in Chapter 2. What connectives are used in compound sentences? Let us first consider a compound sentence of two clauses in two variables, whose connective is "or" ; for example,

$$x + 2y - 5 = 0 \text{ or } 2x + y - 1 = 0 .$$

When is a compound setntence with the connective "or" true? The truth set of this sentence includes all the ordered pairs of numbers which satisfy "$x + 2y - 5 = 0$", as well as all the ordered pairs which satisfy "$2x + y - 1 = 0$", and the graph of its truth set is the pair of lines drawn in Figure 1.

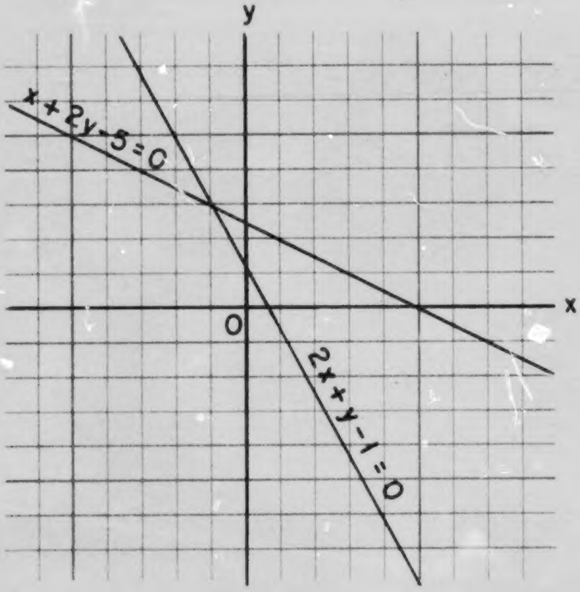


Figure 1

Name three ordered pairs of numbers which belong to the truth set of

$$x + 2y - 5 = 0 .$$

Name four ordered pairs which belong to the truth set of

$$2x + y - 1 = 0 .$$

Which of these ordered pairs of numbers are elements of the truth set of the compound open sentence

$$x + 2y - 5 = 0 \text{ or } 2x + y - 1 = 0 ?$$

If we remember that the sentences "$ab = 0$" and "$a = 0$ or $b = 0$" are equivalent when a and b are real expressions, another way of writing this compound sentence would be

$$(x + 2y - 5)(2x + y - 1) = 0 .$$

Which ordered pairs are elements of the truth set of the compound open sentence "$x + 2y - 5 = 0$ and $2x + y - 1 = 0$" ? Note that only one ordered pair $(-1, 3)$, satisfies both clauses of this sentence, and therefore the graph of the truth set of the open sentence "$x + 2y - 5 = 0$ and $2x + y - 1 = 0$" is the intersection of the pair of lines in Figure 1.

In this chapter we shall devote most of our attention to compound open sentences made up of two clauses connected by and. This sort of compound open sentence, with the connective "and" , is often written

$$\begin{cases} 2x + y - 1 = 0 , \\ x + 2y - 5 = 0 . \end{cases}$$

This is called a system of equations in two variables. When we talk about the truth set of a system of equations we mean the intersection of the truth sets of the two sentences. As we have seen, the truth set of

$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$$

is $\{(-1, 3)\}$.

Exercises 12 - 1a.

1. Find the truth sets of the following systems of equations by drawing the graphs of each pair of open sentences and guessing the coordinates of the intersections. (In each case, verify that your guess satisfies the sentence.)

(a) $\begin{cases} 2x - y - 2 = 0 \\ 3x + y - 3 = 0 \end{cases}$ (c) $\begin{cases} x - 3 = 0 \\ 2x + 3y - 9 = 0 \end{cases}$

(b) $\begin{cases} 2x + 3y - 3 = 0 \\ x + y = 0 \end{cases}$ (d) $\begin{cases} 5x + y - 10 = 0 \\ 2x + 2y - 8 = 0 \end{cases}$

(e) $\begin{cases} 5x - y + 13 = 0 \\ x - 2y - 12 = 0 \end{cases}$

2. Draw the graphs of the truth sets of the following sentences:

(a) $x + 2y - 6 = 0$ or $2x + y - 5 = 0$

(b) $(2x - 3y + 9)(3x + y - 2) = 0$

Did you have trouble guessing the coordinates of the intersection points in problems 1 (d) and (e)? Let us see if we can find a systematic way to obtain the ordered pair without guessing.

Returning to the compound sentence "$x + 2y - 5 = 0$ and $2x + y - 1 = 0$", and looking at Figure 2 , we see that there are many compound open sentences whose truth set is $\{(-1, 3)\}$; for example, "$2x + y - 1 = 0$ and $y - 3 = 0$", and "$x + 2y - 5 = 0$ and $x + 1 = 0$" are two such equivalent

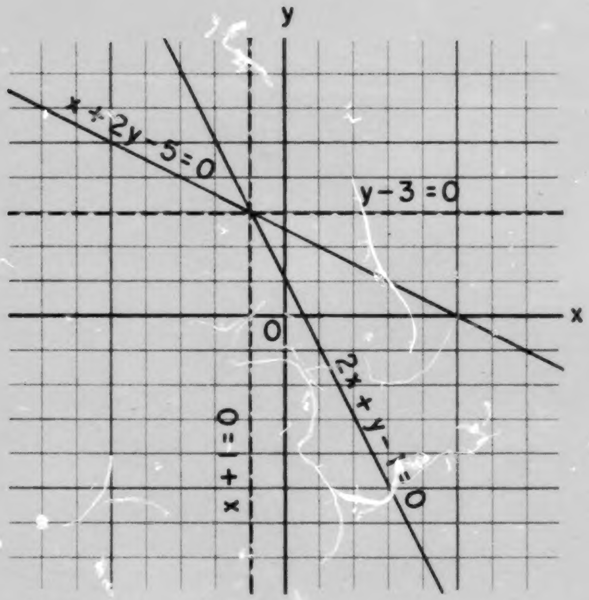


Figure 2

compound sentences, because their graphs are pairs of lines intersecting in $(-1, 3)$. State at least five more compound sentences whose truth set is $\{(-1, 3)\}$. What is the truth set of

$$x + 1 = 0 \quad \text{and} \quad y - 3 = 0 \ ?$$

From this it appears that we could easily find the truth set of any compound open sentence of the type

$$2x + y - 1 = 0 \quad \text{and} \quad x = 2y - 5 = 0$$

if we had a method for getting the equations of the horizontal and vertical lines through the intersection of the graphs of the two clauses.

There are many lines through any point. Here is a method which, as we shall see, will give us the equation of any line through the intersection of two given lines, provided that the lines do intersect in a single point. We shall again use the system

$$\begin{cases} x + 2y - 5 = 0 \\ 2x + y - 1 = 0 \end{cases}$$

to illustrate.

We multiply the expression on the left of the first equation by any number, say 3, and the expression on the left of the second equation by any number, say 7, and form the sentence

$$3(x + 2y - 5) + 7(2x + y - 1) = 0 .$$

We see that:

(1) The coordinates of the point of intersection $(-1, 3)$ of the two lines satisfy this new sentence:

$$3\big(-1 + 2(3) - 5\big) + 7\big(2(-1) + 3 - 1\big) = 3(0) + 7(0) = 0 .$$

In general, we know that a point belongs to the graph of a sentence if its coordinates satisfy the sentence. So the graph of our new open sentence "$3(x + 2y - 5) + 7(2x + y - 1) = 0$" contains the point of intersection of the two lines "$x + 2y - 5 = 0$" and "$2x + y - 1 = 0$".

(2) The graph is a line, because:

$$\begin{aligned} 3(x + 2y - 5) + 7(2x + y - 1) &= 0 , \\ 3x + 6y - 15 + 14x + 7y - 7 &= 0 , \\ 17x + 13y - 16 &= 0 , \end{aligned}$$

and we found, in Chapter 11, that the graph of any equation of the form $Ax + By + C = 0$ is a line, when either A or B is not 0.

Suppose we use this method to find the equation of a line through the intersection of the graphs of the two equations in problem 1(e) of Exercises 12 - 1a:

$$\begin{cases} 5x - y + 13 = 0 , \\ x - 2y - 12 = 0 . \end{cases}$$

If we have no particular line in mind, we can use any multipliers we wish. Let us choose 3 and -2, and form the equation:

$$3(5x - y + 13) + (-2)(x - 2y - 12) = 0.$$

Let us assume that the point (c, d) is the point of intersection of the graphs of these two equations. Since this point (c, d) is on both graphs, we know that

$$5c - d + 13 = 0 \quad \text{and} \quad c - 2d - 12 = 0$$

is a true sentence. Why?

Substituting the ordered pair (c, d) in the left side of our new equation, we obtain

$$3(5c - d + 13) + (-2)(c - 2d - 12) = 3(0) + (-2)(0) = 0 .$$

Hence, we know that if the graphs of the first two equations intersect in a point (c, d), the new line also passes through (c, d), even though we do not know what the point (c, d), is.

In general, we can say:

> If $Ax + By + C = 0$ and $Dx + Ey + F = 0$ are the equations of two lines which intersect in exactly one point, and if a and b are real numbers, then
>
> $$a(Ax + By + C) + b(Dx + Ey + F) = 0$$
>
> is the equation of a line which passes through the point of intersection of the first two lines.

Now that we have a method for finding the equations of lines through the intersection of two given lines, let us see if we can select our multipliers a and b with more care, so that we can get the equations of lines parallel to the axes.

The system

$$\begin{cases} 5x - y + 13 = 0 \\ x - 2y - 12 = 0 \end{cases}$$

gave us some trouble when we tried to guess its truth set from the graph; so let us see if this new approach will help us. Form the sentence

$$a(5x - y + 13) + b(x - 2y - 12) = 0.$$

Let us choose a as 2 and b as -1, so that the coefficients of y become opposites:

$$\begin{aligned} (2)(5x - y + 13) + (-1)(x - 2y - 12) &= 0 , \\ 10x - 2y + 26 - x + 2y + 12 &= 0 , \\ 9x + 38 &= 0 , \\ x + 4\frac{2}{9} &= 0 , \end{aligned}$$

This last equation represents the line through the intersection of the graphs of the two given equations and parallel to the y-axis. Let us go back and select new multipliers that will give us the equation of the line through the intersection point and parallel to the x-axis. What multipliers shall we use? Since we want the coefficients of x to be opposites we choose $a = 1$ and $b = -5$?

$$\begin{aligned} (1)(5x - y + 13) + (-5)(x = 2y - 12) &= 0 , \\ 5x - y + 13 - 5x + 10y + 60 &= 0 , \\ 9y + 73 &= 0 , \\ y + 8\frac{1}{9} &= 0 . \end{aligned}$$

We now have the equations of two new lines, "$x + 4\frac{2}{9} = 0$" and "$y + 8\frac{1}{9} = 0$", each of which passes through the point of intersection of the graphs of the first two equations. Why? This reduces our original problem to finding the point of intersection of these new lines. Can you see what it is? So we see that the truth set of the system

$$\begin{cases} 5x - y + 13 = 0 \\ x - 2y - 12 = 0 \end{cases}$$

is $\left\{\left(-4\frac{2}{9}, -8\frac{1}{9}\right)\right\}$. Verify this fact by showing that these coordinates satisfy both equations.

Now we have a procedure for solving any system of two linear equations. We choose multipliers so as to obtain an equivalent system of lines which are horizontal and vertical. The choice of the multipliers will become easy with practice.

Consider another example: Three times the smaller of two numbers is 6 greater than twice the larger, and three times the larger is 7 greater than four times the smaller. What are the numbers?

The smaller number x and the larger y must satisfy the open sentence

$$3x - 2y - 6 = 0 \text{ and } 4x - 3y + 7 = 0,$$

Choose multipliers so that the coefficients of x will be opposites. 4 and -3 will do the trick:

$$4(3x - 2y - 6) + (-3)(4x - 3y + 7) = 0,$$

$$12x - 8y - 24 - 12x + 9y - 21 = 0,$$

$$y - 45 = 0.$$

Now we could choose multipliers so that the coefficients of y would be opposites. Another way to find the line through the intersection and parallel to the y-axis is as follows: On the line "$y - 45 = 0$" every point has ordinate 45 . Thus, the ordinate of the point of intersection is 45 . The solution of the sentence "$3x - 2y - 6 = 0$" with ordinate 45 is obtained by solving "$3x - 2(45) - 6 = 0$" or its equivalent, "$x - 32 = 0$" . Hence, the sentence "$3x - 2y - 6 = 0$ and $4x - 3x + 7 = 0$" is equivalent to the sentence "$y - 45 = 0$ and $x - 32 = 0$". Now it is easy to read off the solution of the system as (32, 45) .

In the above example, what is the solution of the sentence "$4x - 3y + 7 = 0$" with ordinate 45 ? Does it matter in which sentence we assign y as 45 ?

Exercises 12 - 1b

1. Find the truth sets of the following systems of equations by the method just developed. Draw the graphs of each pair of equations in (a) and (b) with reference to a different set of axes.

(a) $\begin{cases} 3x - 2y - 14 = 0 \\ 2x + 3y + 8 = 0 \end{cases}$

(b) $\begin{cases} 5x + 2y = 4 \\ 3x - 2y = 12 \end{cases}$

(c) $\begin{cases} 5x - y = 32 \\ x - 2y - 19 = 0 \end{cases}$

(d) $\begin{cases} 3x - 2y = -18 \\ 2x - 7y = 34 \end{cases}$

(e) $\begin{cases} x + y - 30 = 0 \\ x - y + 7 = 0 \end{cases}$

(f) $\begin{cases} y = 7x + 5 \\ 4x = y - 3 \end{cases}$

2. We can also use the operations on equations stated in Chapter 10 to solve a system of equations. The method which results is essentially the same as that used above. For example, consider the system:

$$\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 8 = 0 \end{cases}$$

and assume that (c, d) is a solution of the system. Then each of the following equations is true (Give reasons why each is true.) :

$$\begin{aligned} 3c - 2d - 5 &= 0 , \\ c + 3d - 8 &= 0 ; \end{aligned}$$

$$\begin{aligned} 3(3c - 2d - 5) &= 3(0) , \\ 2(c + 3d - 8) &= 2(0) ; \end{aligned}$$

$$\begin{aligned} 9c - 6d - 15 &= 0 , \\ 2x + 6d - 16 &= 0 ; \end{aligned}$$

$$\begin{aligned} 11c - 31 &= 0 , \\ c &= \frac{31}{11} . \end{aligned}$$

Also,

$$\begin{aligned} 3c - 2d - 5 &= 0 , \\ -3(c + 3d - 8) &= -3(0) , \\ -3c - 9d + 24 &= 0 ; \end{aligned}$$

$$\begin{aligned} -11d + 19 &= 0 , \\ d &= \frac{19}{11} . \end{aligned}$$

So _if_ there is a solution of the system

$$\begin{cases} 3x - 2y - 5 = 0 , \\ x + 3y - 8 = 0 , \end{cases}$$

then that solution is $(\frac{31}{11} , \frac{19}{11})$. We must verify that this is a solution.

$$3(\frac{31}{11}) - 2(\frac{19}{11}) - 5 = 0 ,$$

$$\frac{31}{11} + 3(\frac{19}{11}8 - 8 = 0 .$$

Are these sentences true?

This is often called the _addition_ _method_ of solving systems of equations. Use this method for finding the truth sets of the following systems:

(a) $\begin{cases} x - 4y - 6 = 0 \\ 3x + 5y + 1 = 0 \end{cases}$ (c) $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 2y \end{cases}$

(b) $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 2x \end{cases}$ (d) $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 3x \end{cases}$

3. Translate each of the following into open sentences with two variables. Find the truth set of each.

(a) Three hundred eleven tickets were sold for a basketball game, some for pupils and some for adults. Pupil tickets sold for 25 cents each and adult tickets for 75 cents each. The total money received was $108.75. How many pupil and how many adult tickets were sold?

(b) The Boxer family is coming to visit, and no one knows how many children they have. Elsie, one of the girls, says she has as many brothers as sisters; her brother Jimmie says he has twice as many sisters as brothers. How many boys and how many girls are there in the Boxer family?

(c) A home room bought three-cent and four-cent stamps to mail bulletins to the parents. The total cost was $12.67. If they bought 352 stamps, how many of each kind were there?

(d) A bank teller has 154 bills of one-dollar and five-dollar denominations. He thinks his total is $466. Has he counted his money correctly?

4. Find the truth sets of the following compound open sentences. Draw the graphs. Do they help you with (b) and (c) ?

   (a) $x - 2y + 6 = 0$ and $2x + 3y + 5 = 0$.

   (b) $2x - y - 5 = 0$ and $4x - 2y - 10 = 0$.

   (c) $2x + y - 4 = 0$ and $2x + y - 2 = 0$.

5. Find the equation of the line through the intersection of the lines $5x - 7y - 3 = 0$ and $3x - 6y + 5 = 0$ and passing through the origin. (Hint: What is the value of C so that $Ax + By + C = 0$ is a line through the origin?)

In problem 4 you found some compound open sentences whose truth sets were not single ordered pairs of numbers. Which ones were they? Let us look more closely at each of them.

In the open sentence

$$"2x - y - 5 = 0 \text{ and } 4x - 2y - 10 = 0",$$

we note that "$4x - 2y - 10 = 0$" is equivalent to

$$2(2x - y - 5) = 2(0) ;$$

so we see that the graphs of both clauses are identical, as shown in Figure 3, and the lines have many points in common. State some of the numbers of the truth set of the compound sentence. Is the truth set a finite set? What happened when

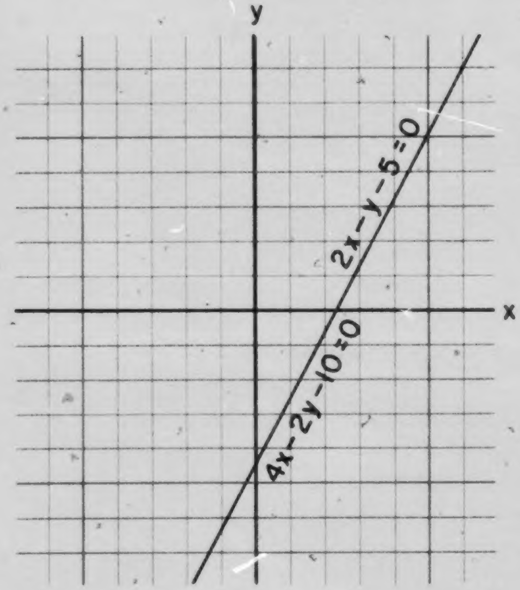


Figure 3

you tried to solve the open sentence algebraically? Why didn't our method work?

A somewhat different condition exists in the compound sentence $2x + y - 4 = 0$ and $2x + y - 2 = 0$. Putting each clause into the y- form, we have

$$y = -2x + 4 \quad \text{and} \quad y = -2x = 2 .$$

What is the slope of the graph of each of these equations? What is the y-intercept nember? We see that the graphs are two parallel lines, as in Figure 4, and there is no intersection point. In such a case, the truth set of the compound sentence is the null set. What happens if we try to solve the sentence algebraically? Why?

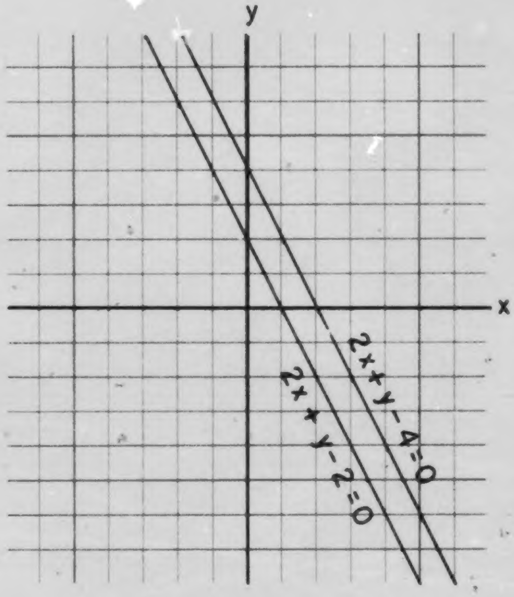


Figure 4.

Let us try to summarize what we have noted here: The truth set of a compound open sentence in two variables, with connective "and", may consist of one ordered pair, many ordered pairs, or no ordered pairs. Correspondingly, the graphs of the two clauses of the open sentence may have one intersection, many intersections, or no intersections.

	Equations	Graphs
Example 1.	$2x - 3y = 4$ and $x + y = 7$; $y = \frac{2}{3}x - \frac{4}{3}$ and $y = -x + 7$. The truth set is $\{(5, 2)\}$.	The two lines which are the graphs of the clauses have one intersection, since the slopes of the lines are not the same. The graph of the truth set is the single point (5, 2).
Example 2.	$2x - 3y = 7$ and $4x - 6y = 14$; $y = \frac{2}{3}x - \frac{7}{3}$ and $y = \frac{4}{6}x - \frac{14}{6}$. The truth set is made up of all the ordered pairs whose coordinates satisfy the first equation. (Note that the second clause is obtained if each side of the first clause of the original open sentence is multiplied by 2.)	The graphs of the two clauses of the open sentence coincide, since the lines have the same slope and the same y-intercept number. The entire line is the graph of the truth set.
Example 3.	$2x - 3y = 7$ and $4x - 6y = 3$; $y = \frac{2}{3}x - \frac{7}{3}$ and $y = \frac{4}{6}x - \frac{3}{6}$. The truth set is $\emptyset$.	The graphs of the two clauses of the open sentence are parallel lines, because they have the same slope but different y-intercept numbers. The graph of the truth set contains no points.

Notice, in Example 3, that the coefficients of x and y in the equation $2x - 3y = 7$ are related in a simple way to those in the equation $4x - 6y = 3$:

$$2 = \frac{1}{2}(4) \quad \text{and} \quad -3 = \frac{1}{2}(-6) .$$

In general, the real numbers a and b are said to be proportional to the real numbers A and B if there is a real number k such that

$$a = kA \quad \text{and} \quad b = kB .$$

Thus, the numbers 2 and -3 are proportional to 4 and -6 , the number k being $\frac{1}{2}$. If two lines are parallel, what can you say about the coefficients of x and y in their equations?

Exercises 12 - 1c.

1. Draw the graphs of the open sentences in Examples 1 to 3 above. Find the truth set of Example 1 algebraically.

2. Solve the following compound open sentences. Draw the graphs in (a) and (b) .

   (a) $3x + 4y - 13 = 0$ and $5x - 2y + 13 = 0$.

   (b) $x + 5y - 17 = 0$ and $2x - 3y - 8 = 0$.

   (c) $5x - 4y + 2 = 0$ and $10x - 8y + 4 = 0$.

   (d) $12x - 4y - 5 = 0$ and $6x + 8y - 5 = 0$.

   (e) $x - 2y - 4 = 0$ and $3x - 6y - 12 = 0$.

   (f) $3(5x - 2y) - 1 = 0$ and $4(7x + 2y) + 2(5x - 3y) = 0$.

   (g) $\frac{1}{3}(\frac{6x}{7} - \frac{3y}{5}) - 1 = 0$ and $\frac{2}{3}(\frac{4x}{7} + \frac{y}{10}) - \frac{7}{3} = 0$.

3. Consider the system,

$$\begin{cases} 2x - y - 7 = 0 \,, \\ 5x + 2y - 4 = 0. \end{cases}$$

Suppose we write the y-form for each equation and draw its graph

$$y = 2x - 7 \quad \text{and} \quad y = \frac{5}{2}x + 2 \,.$$

At what point on the graph of this system are the values of y equal? What is the value of x at this point? If we set the values of y in the two sentences equal to each other, we have the open sentence in one variable,

$$2x - 7 = -\frac{5}{2}x + w \,.$$

The truth set of this sentence is {2}. Using this value for x in each open sentence which is in the y-form we get:

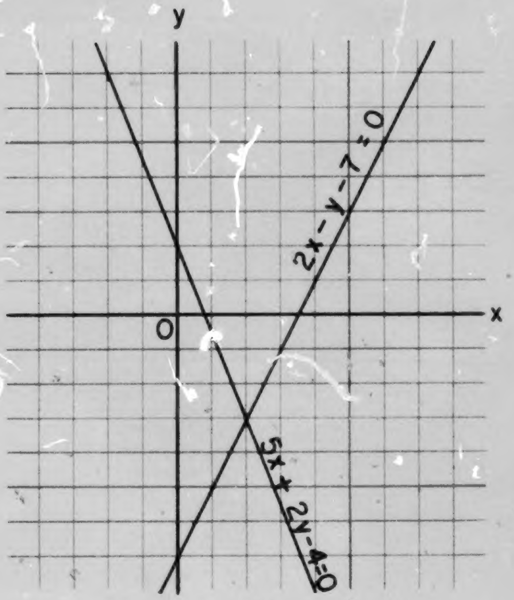


Figure 5.

$$y = 2(2) - 7 \qquad\qquad y = -\frac{5}{2}(2) + 2$$

$$y = -3 \qquad\qquad y = -3$$

Why do we get "y = -3" in both cases? Hence, if the compound open sentence "2x - y - 7 = 0 and 5x + 2y - 4 = 0" has a solution, it must be (2, -3). Verify that (2, -3) is the solution.

Suppose that we shorten our work somewhat by writing only the first equation in its y-form.

$$y = 2x - 7 ,$$

and then substituting the expression "$2x - 7$" for y in the second equation:

$$5x = 2(2x - 7) - 4 = 0 .$$

Let us proceed to solve this open sentence in one variable

$$5x + 4x - 14 - 4 = 0 ,$$

$$9x - 18 = 0 ,$$

$$x = 2 .$$

Then,

$$y = 2x - 7 = 2(2) - 7 = -3$$

so that $(2, -3)$ is the possible solution of the system

$$\begin{cases} 2x - y - 7 = 0 , \\ 5x + 2y - 4 = 0 . \end{cases}$$

The method just described in which we "solve one of the equations for y in terms of x" and then substitute this expression for y into the other equation is called a substitution method. Use this method to solve the following systems:

(a) $\begin{cases} 5x + 2y - 4 = 0 , \\ 3x - 2y - 4 = 0 . \end{cases}$ (d) $\begin{cases} 3x + y + 18 = 0 , \\ 2x - 7y - 34 = 0 . \end{cases}$

(b) $\begin{cases} 5x + 2y - 4 = 0 , \\ 10x + 4y - 8 = 0 . \end{cases}$ (e) $\begin{cases} y = \frac{3}{2}x + 4 , \\ y = -\frac{3}{2}x \end{cases}$

(c) $\begin{cases} 3x - 4y - 1 = 0 , \\ 7x + 4y - 9 = 0 . \end{cases}$

4. As we have seen, the truth set of the compound open sentence

$$Ax + By + C = 0 \text{ and } Dx + Ey + F = 0$$

may consist of one ordered pair of numbers, of many ordered pairs, or of no ordered pairs.

(a) If the truth set consists of one ordered pair, what can you say about the graphs of the clauses?

(b) If the truth set consists of many ordered pairs, what is true of the graphs of the two clauses? Are the two clauses of the compound sentence equivalent?

(c) If the truth set is $\emptyset$, how are the coefficients of x and y related in the two clauses? What is true of the graphs of the clauses?

5. Consider the system:

$$\begin{cases} 4x + 2y - 11 = 0 , \\ 3x - y - 2 = 0 . \end{cases}$$

Draw the graph of each equation. Put each equation in the "x-form". At what point on the graph of the system are the values of x equal? What is the value of y at this point?

Write an open sentence in one variable which has this value of y as its truth set. This is also called a substitution method. Use this substitution method involving x-forms to find the truth sets of the following systems?

(a) $\begin{cases} 3x + 2y = 1 , \\ 2x - 3y = 18 . \end{cases}$

(c) $\begin{cases} x = 2y - \frac{1}{6} , \\ 2x + y = 1\frac{1}{3} \end{cases}$

(b) $\begin{cases} x - 2y = 0 , \\ x + 2y = 0 . \end{cases}$

(d) $\begin{cases} 5 - (x + y) = 2y , \\ 2x - (3y + 1) = 1 . \end{cases}$

6. Translate each of the following into open sentences with two variables. Find the truth set of each, and answer the question asked.

   (a) A 90 % solution of alcohol is to be mixed with a 75 % solution to make 20 quarts of a 78 % solution. How many quarts of the 90 % solution should be used?

   (b) A dealer has some cashew nuts that sell at $ 1.20 a pound and almonds that sell at $ 1.50 a pound. How many pounds of each should he put into a mixture of 200 pounds to sell at $ 1.32 a pound?

   (c) A and B are 30 miles apart. If they leave at the same time and travel in the same direction, A overtakes B in 60 hours. If they walk toward each other they meet in 5 hours. What are their speeds?

   (d) It takes a boat $1\frac{1}{2}$ hours to go 12 miles down stream, and 6 hours to return. Find the speed of the current and the speed of the boat in still water.

   (e) Hugh weighs 80 pounds and Fred weighs 100 pounds. They balance on a teeterboard that is 9 feet long. Each sits at an end of the board. How far is each boy from the fulcrum?

   (f) Three pounds of apples and four pounds of bananas cost $1.08, while 4 pounds of apples and 3 pounds of of bananas cost $ 1.02. What is the price per pound of apples? Of bananas?

(g) In a 300 mile race the driver of car A gives the driver of car B a start of 25 miles, and still finishes one-half hour sooner. In a second trial, the driver of car A gives the driver of car B a start of 60 miles and loses by 12 minutes. What were the average speeds of cars A and B in miles per hour?

12 - 2. Systems of inequalities. In 12 - 1 we defined a system of equations as a compound open sentence in which two equations are joined by the connective "and". We also introduced a notation for this. Carrying the idea over to inequalities, let us consider systems like the following:

(a) $\begin{cases} x + 2y - 4 > 0 , \\ 2x - y - 3 > 0 . \end{cases}$ (b) $\begin{cases} 3x - 2y - 5 = 0 , \\ x + 3y - 9 \leq 0 . \end{cases}$

(a) What would the graph of $x + 2y - 4 > 0$ look like? You recall that we first draw the graph of

$$x + 2y - 4 = 0 ,$$

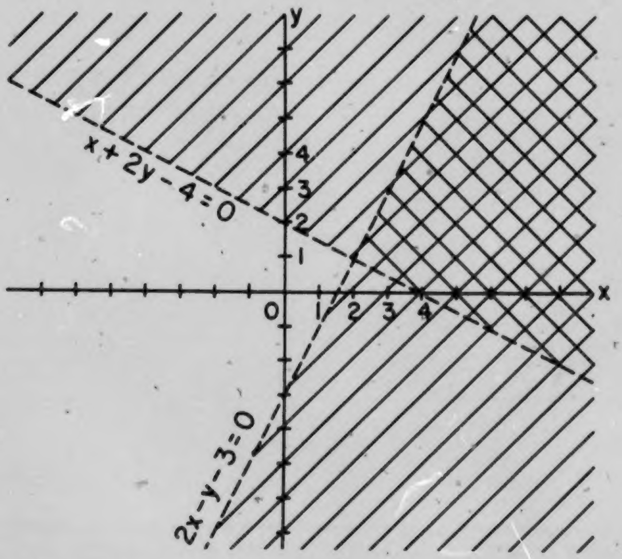


Figure 6

using a dashed line along the boundary. Why? Then we shade the region above the line, since the graph of "$x + 2y - 4 > 0$", i.e. of "$y > -\frac{1}{2}x + 2$" : consists of all those points whose ordinate is greater than "two more than $-\frac{1}{2}$ times the abscissa". In a similar way, we shade the region where "$y < 2x - 3$". Why? This is the region below the line whose equation is "$2x - y - 3 = 0$". Why is the line here also dashed? When would we use a solid line as boundary?

Since the truth set of a compound open sentence with the connective and is the intersection of the truth sets of the two clauses, it follows that the truth set of the system (a) is the region indicated by criss-cross shading in Figure 6.

(b) What would be the graph of a system in which we have one equation and one inequality, such as Example (b)? What is the graph of "$3x - 2y - 5 = 0$"?

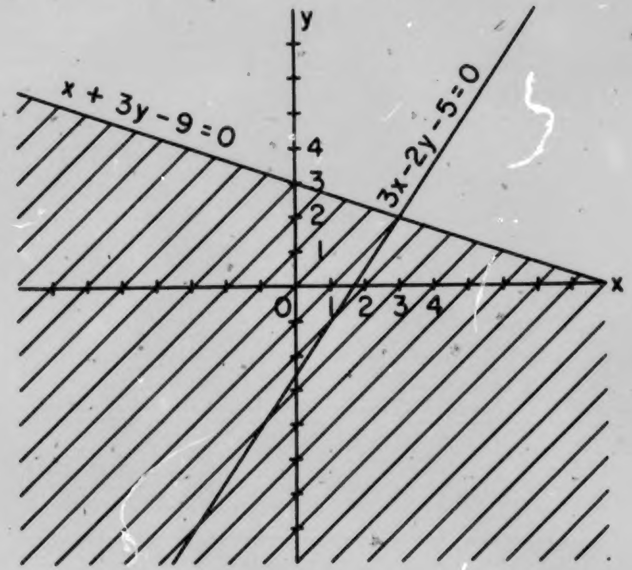


Figure 7

Is the graph of "$x + 3y - 9 \leq 0$" the region above or below the line

"$x + 3y - 9 = 0$" ?

Is the line included? Study Figure 7 carefully, and describe the graph of the system

$$\begin{cases} 3x - 2y - 5 = 0 , \\ x + 3y - 9 \leq 0 . \end{cases}$$

Exercises 12 - 2a.

Draw graphs of the truth sets of the following systems:

1. $\begin{cases} 2x + y > 8 \\ 4x - 2y \leq 4 \end{cases}$

2. $\begin{cases} 6x + 3y < 0 \\ 4x - y < 6 \end{cases}$

3. $\begin{cases} 5x + 2y + 1 > 0 \\ 3x - y - 6 = 0 \end{cases}$

4. $\begin{cases} 4x + 2y = -1 \\ y - x \geq 4 \end{cases}$

5. $\begin{cases} 2x + y < 4 \\ 2x + y > 6 \end{cases}$

6. $\begin{cases} 2x + y > 4 \\ 2x + y < 6 \end{cases}$

7. $\begin{cases} 2x - y \leq 4 \\ 4x - 2y < 8 \end{cases}$

Let us consider the graph of the compound open sentence

$$x - y - 2 > 0 \text{ or } x + y - 2 > 0 .$$

First we draw the graphs of the clauses "$x - y - 2 > 0$" and "$x + y - 2 > 0$" .

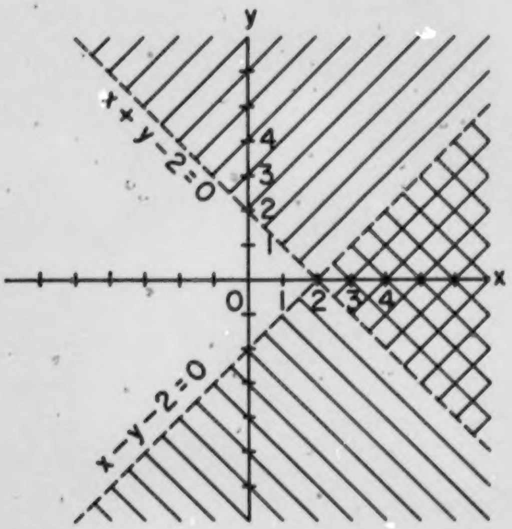


Figure 8

Next we recall that the truth set of a compound open sentence with the connective or is the union of the truth sets of the clauses. Hence, the graph of the compound open sentence under consideration includes the entire shaded region in Figure 8.

Exercises 12 - 2b.

Draw the graphs of the truth sets of the following sentences:

1. $2x + y + 3 > 0$ or $3x + y + 1 < 0$.
2. $2x + y + 3 < 0$ or $3x - y + 1 < 0$.
3. $2x + y + 3 \leq 0$ or $3x + y + 1 \geq 0$.
4. $2x + y + 3 > 0$ and $3x = y + 1 < 0$.

To complete the picture, let us consider the compound open sentence:

$$(x - y - 2)(x + y - 2) > 0 .$$

Remember that "$ab > 0$" means that "the product of a and b is a positive number". What can be said of a and b if $ab > 0$? Thus we have the two possibilities:

$$x - y - 2 > 0 \quad \text{and} \quad x + y - 2 > 0 ,$$

or

$$x - y - 2 < 0 \quad \text{and} \quad x + y - 2 < 0 .$$

In Figure 8, the graph of "$x - y - 2 > 0$ and $x + y - 2 > 0$" is the region indicated by the criss-cross shading, while the graph of "$x - y - 2 < 0$ and $x + y - 2 < 0$" is the unshaded region. So the graph of

$$(x - y - 2)(x + y - 2) > 0$$

consists of all the points in these two regions of the plane.

Which areas form the graph of the open sentence

$$(x - y - 2)(x + y - 2) < 0 \ ?$$

(If $ab < 0$, what can be said of a and b ?)

To summarize, we list the following pairs of equivalent sentences (a and b are real expressions):

$ab = 0$: $a = 0$ or $b = 0$.

$ab > 0$: $a > 0$ and $b > 0$, or $a < 0$ and $b < 0$.

$ab < 0$: $a > 0$ and $b < 0$, or $a < 0$ and $b > 0$.

Verify these equivalences by going back to the definition of the product of real numbers.

Exercises 12 - 2c.

1. Draw the graphs of the truth sets of the following open sentences
   (a) $(2x - y - 2)(3x + y - 3) > 0$.
   (b) $(x + 2y - 4)(2x - y - 3) < 0$.
   (c) $(x + 2y - 6)(2x + 4y + 4) > 0$.
   (d) $(x - y - 3)(3x - 3y - 9) < 0$.

2. Draw the graphs of the truth sets of the following open sentences
   (a) $x - 3y - 6 = 0$ and $3x + y + 2 = 0$.
   (b) $(x - 3y - 6)(3x + y + 2) = 0$.
   (c) $x - 3y - 6 > 0$ and $3x + y + 2 > 0$.
   (d) $x - 3y - 6 < 0$ and $3x + y + 2 < 0$.
   (e) $x - 3y - 6 > 0$ and $3x + y + 2 = 0$.
   (f) $x - 3y - 6 < 0$ or $3x + y + 2 < 0$.
   (g) $x - 3y - 6 = 0$ or $3x + y + 2 \geq 0$.
   (h) $(x - 3y - 6)(3x + y + 2) > 0$.
   (i) $(x - 3y - 6)(3x + y + 2) < 0$.

3. Draw the graph of the truth set of each of these systems of inequalities: (The brace again indicates a compound sentence with connective and.)

(a) $$\begin{cases} x > 0 , \\ y \geq 0 , \\ 3x + 4y \leq 12 . \end{cases}$$

(c) $$\begin{cases} -4 < x < 4 , \\ -3 < y < 3 . \end{cases}$$

(b) $$\begin{cases} y \geq 2 \\ 4y \leq 3x + 8 , \\ 4y + 5x \leq 40 . \end{cases}$$

4. (Optional) A football team finds itself on its own 40 yard line, in possession of the ball, with five minutes left in the game. The score is 3 to 0 in favor of the opposing team. The quarterback knows the team should make 3 yards on each running play, but will use 30 seconds per play. He can make 20 yards on a successful pass play, which uses 15 seconds. However, he usually completes only one pass out of three. What combination of plays will assure a victory, or what should be the strategy of the quarterback?

Chapter 13

Quadratic Polynomials

13 - 1. *Graphs of quadratic polynomials*. In Chapter 9 we first studied *quadratic polynomials*, that is, polynomials in one variable which involve the square but no higher powers of the variable. Every such polynomial can be written in the form

$$Ax^2 + Bx + C,$$

where A, B, and C are real numbers with $A \neq 0$. Is "$-2(x + 1)^2 + 3$" a quadratic polynomial? By the *graph of the polynomial* $Ax^2 + Bx + C$ we mean the graph of the open sentence

$$y = Ax^2 + Bx + C.$$

We can make an approximate drawing of the graph of a quadratic polynomial by locating some of the points of the graph.

Example 1. Draw the graph of the polynomial

$$x^2 - 2x - 3.$$

Let us list some ordered pairs satisfying the equation

$$y = x^2 - 2x - 3.$$

x	-2	$-\frac{3}{2}$	-1	$-\frac{2}{3}$	0	$\frac{1}{2}$	1	$\frac{4}{3}$		$\frac{5}{2}$	3	4
y	5	$\frac{9}{4}$			-3		-4		-3	$-\frac{7}{4}$		

Fill in the missing numbers in this table and then locate these points with reference to a set of coordinate axes. The arrangement of the points suggests that the graph might look like the one sketched in Figure 1.

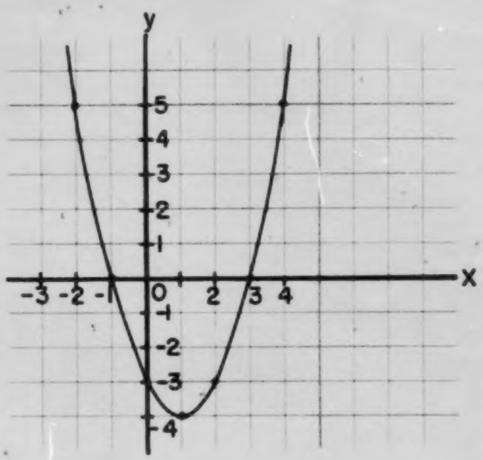


Figure 1

By locating more points whose coordinates staisfy the equation you can convince yourself that the graph does indeed have the indicated shape. A more systematic discussion of the shape of such graphs will be found in Chapter 14.

Exercises 13 - 1a.

Draw, approximately, the graphs of the polynomials:

1. $2x^2$, for x between -2 and 2.
2. $x^2 - 2$, for x between -3 and 3.
3. $-\frac{1}{2}x^2 + x$, for x between -3 and 3.
4. $x^2 + x + 1$, for x between -3 and 2.
5. $x^2 - 4x + 4$, for x between -1 and 5.
6. $2x^2 - 3x - 5$, for x between -2 and 3.

You probably noticed that the preceding problems took a good deal of time and effort. Even then, you were only guessing at the shapes of the graphs. Let us try to develop a more precise method for drawing such graphs.

Start with the most simple quadratic polynomial, " x^2 ". (In previous sections we located some points on the graph of "$y = x^2$".) Then let us see how this graph differs from that of "$y = \frac{1}{2}x^2$", of "$y = 2x^2$", of "$y = -\frac{1}{2}x^2$". In general, what will be the shape of the graph of

$$y = ax^2 ,$$

where a is a non-zero real number? If we draw all these graphs with reference to one set of axes, we will be able to compare them. A list of values of these polynomials for given values of x is as follows:

x	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	-0.1	0	$\frac{1}{3}$	1	$\frac{4}{3}$	2	
x^2		4		1	$\frac{1}{4}$		0		1		4	
$2x^2$		8		2	$\frac{1}{2}$		0		2		8	18
$\frac{1}{2}x^2$		2		$\frac{1}{2}$	$\frac{1}{8}$		0		$\frac{1}{2}$		2	
$-\frac{1}{2}x^2$		-2		$-\frac{1}{2}$	$\frac{1}{8}$		0		$-\frac{1}{2}$		-2	

You fill in the missing numbers. The graphs in Figure 2 are suggested by this table.

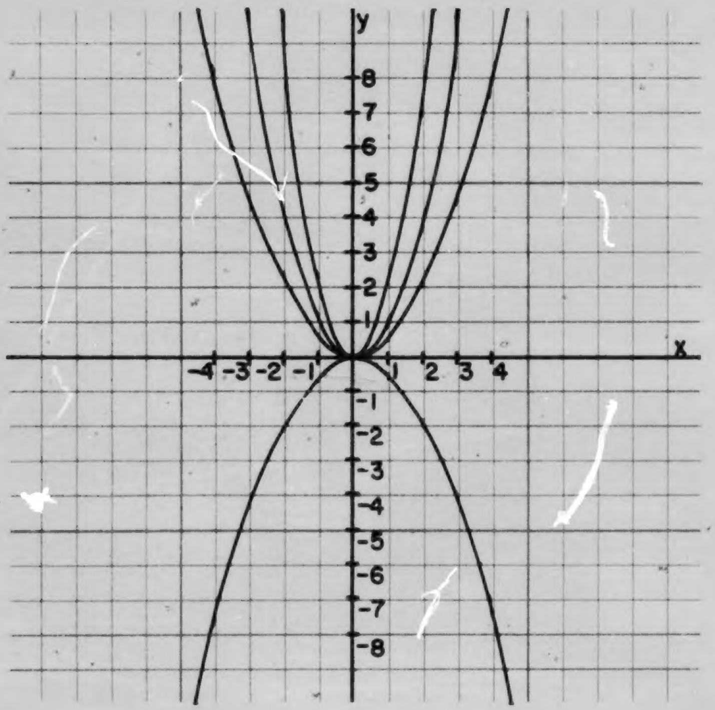


Figure 2

Exercises 13 . 1b.

1. How could you obtain the graph of " $2x^2$ " from the graph of " x^2 " ?
2. How could you obtain the graph of " $-\frac{1}{2}x^2$ " from the graph of " $\frac{1}{2}x^2$ " ?
3. Draw the graph of " $5x^2$ " for x between -1 and 1.
4. Draw the graph of " $\frac{1}{5}x^2$ " for x between -10 and 10.
5. How could you obtain the graph of " $-10x^2$ " from the graph of " $10x^2$ " ?
6. Explain how you could obtain the graph of " $-ax^2$ " from the graph of " ax^2 ", where a is any non-zero real number.

Now that we have a graph of the polynomial " ax^2 ", for any non-zero number a, let us move this graph horizontally to obtain graphs of other quadratic polynomials. As an example, let us draw the graph of

$$\frac{1}{2}(x - 3)^2$$

and see how it can be obtained from the graph of " $\frac{1}{2}x^2$ ". Let us list a table of coordinates satisfying the equation

$$y = \frac{1}{2}(x - 3)^2 .$$

x	0	$\frac{1}{3}$	1	$\frac{3}{2}$	2	2.5		$\frac{13}{4}$	4	5
y		$\frac{32}{9}$		$\frac{9}{8}$			0	$\frac{1}{32}$		

Fill in the missing numbers. The graph is compared in Figure 3 with the graph of " $y = \frac{1}{2}x^2$ ".

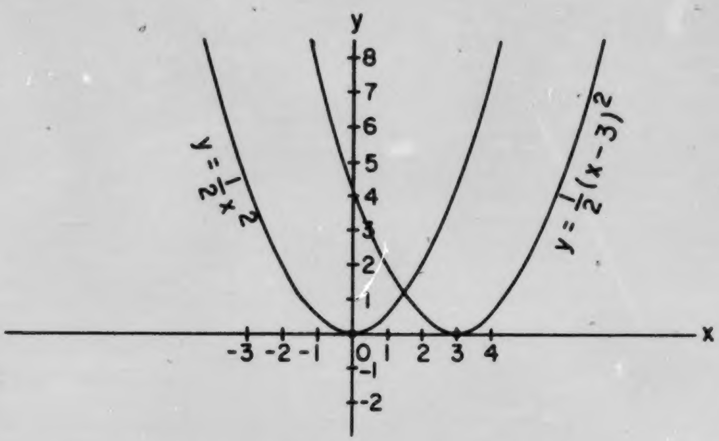


Figure 3

We notice that the graph of $y = \frac{1}{2}(x - 3)^2$ has exactly the same shape as the graph of

$$y = \frac{1}{2}x^2 ,$$

but is 3 units to the right. In the same way we could verify that the graph of " $y = 2(x + 2)^2$ " is 2 units to the left of the graph of " $y = 2x^2$ " and has the same shape as " $y = 2x^2$ ". How could we obtain the graph of

$$y = - (x + 3)^2$$

from the graph of

$$y = - x^2 \quad ?$$

Exercise 13 - 1c.

1. After setting up a table of coordinates of points, draw carefully the graph of

$$y = 2(x + 2)^2 ,$$

with reference to the same coordinate axes draw the graph of

$$y = 2x^2 .$$

From the figure describe how you could obtain the graph of " $y = 2(x + 2)^2$ " from the graph of " $y = 2x^2$ " .

2. For each of the following, describe how you could obtain the graph of the first from the graph of the second equation:

(a) $y = 3(x + 4)^2$; $y = 3x^2$

(b) $y = -2(x - 3)^2$; $y = -2x^2$

(c) $y = - \frac{1}{2}(x + 1)^2$; $y = - \frac{1}{2}x^2$

(d) $y = \frac{1}{3}(x + \frac{1}{2})^2$; $y = \frac{1}{3}x^2$

3. Give a general rule for obtaining the graph of " $y = a(x - h)^2$ " from the graph of " $y = ax^2$ " , where a and h are real numbers and $a \neq 0$.

Next, let us move the graphs of polynomials vertically. Consider the quadratic polynomial,

$$\frac{1}{2}(x - 3)^2 + 2 ,$$

and compare it with the graph, which we have already obtained, of " $\frac{1}{2}(x - 3)^2$ " . A table of coordinates satisfying " $y = \frac{1}{2}(x - 3)^2 + 2$ " is the following:

x	0	$\frac{1}{3}$	1	$\frac{3}{2}$	2	2.5	3	$\frac{13}{4}$	4	5
y		$\frac{50}{9}$		$\frac{25}{8}$			2	$\frac{65}{32}$		4

(You have probably observed that each ordinate in this table is 2 greater than the cooresponding ordinate in the preceding table.)

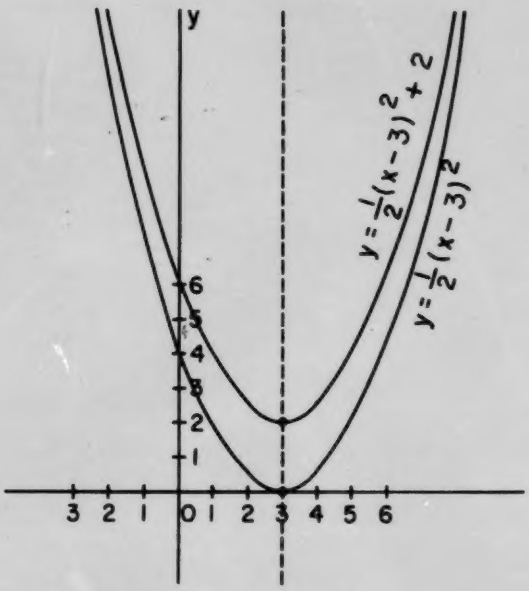


Figure 4

Again we observe that the shape of the graph has not changed, but that the graph of

$$y = \frac{1}{2}(x - 3)^2 + 2$$

is obtained by moving the graph of

$$y = \frac{1}{2}(x - 3)^2$$

upward 2 units. Similarly, we can show that the graph of " $y = 2(x + 2)^2 - 3$ " can be obtained by moving the graph of " $y = 2(x + 2)^2$ " downward 3 units.

Finally, we notice that the graphs in Figures 3 and 4 are exactly the same shape, and that we can obtain the graph of " $y = \frac{1}{2}(x - 3)^2 + 2$ " by moving the graph of " $y = \frac{1}{2}x^2$ " to the right 3 units and upward 2 units.

We shall see in Chapter 14 that it is always possible to obtain the graph of

$$y = a(x - h)^2 + k$$

from the graph of

$$y = ax^2$$

by moving the graph of " $y = ax^2$ " horizontally h units and vertically k units.

These graphs (of quadratic polynomials) are called parabolas. The lowest (or highest) point on the graph is called the vertex, and the vertical line through the vertex is called the axis. Thus, the vertex of the parabola whose equation is

$$y = 2x^2$$

is (0, 0) and its axis is the line with the equation $x = 0$. What are the vertex and axis of the parabola whose equation is

$$y = \frac{1}{2}(x - 3)^2 + 2 \text{ ?}$$

Exercises 13 - 1d.

1. Describe how the graphs of "$y = x^2 - 3$" and "$y = x^2 + 3$" could be obtained from the graph of "$y = x^2$"? Draw all three graphs with reference to the same axes.
2. How could the graph of "$y = 2(x - 2)^2 + 3$" be obtained from the graph of "$y = 2x^2$"? Draw both graphs with reference to the same axes.
3. Draw the parabola whose equation is "$y = (x + 1)^2 - \frac{1}{2}$". Describe how you obtain this graph from the graph of "$y = x^2$". What are the coordinates of its vertex and the equation of its axis?
4. Draw the parabola whose equation is "$y = -2(x + \frac{1}{2})^2 + 3$". How can this parabola be obtained from the graph of "$y = -2x^2$"?
5. Find equations for the following parabolas:
   (a) The graph of "$y = x^2$" moved 5 units to the left and 2 units downward.
   (b) The graph of "$y = -x^2$" moved 2 units to the left and 3 units upward.
   (c) The graph of "$y = \frac{1}{3}x^2$" moved $\frac{1}{2}$ unit to the right and 1 unit downward.
   (d) The graph of "$y = \frac{1}{2}(x + 7)^2 - 4$" moved 7 units to the right and 4 units upward.

6. Describe, without drawing, the graph of each of the following:

(a) $y = 3(x - 2)^2 - 4$

(b) $y = -(x + 3)^2 + 1$

(c) $y = \frac{1}{2}(x - 2)^2 - 2$

(d) $y = -2(x + 1)^2 + 2$

13 - 2. Standard Forms. We have learned how to obtain the graph of "$y = (x - 1)^2 - 4$" quickly. This is the parabola obtained by moving the graph of "$y = x^2$" 1 unit to the right and 4 units downward. We also notice that $(x - 1)^2 - 4 = x^2 - 2x - 3$, for every real number x. Therefore, we have obtained the graph of the equation

$$y = x^2 - 2x - 3 .$$

Suppose we were given the equation in the form "$y = x^2 - 2x - 3$" instead of "$y = (x - 1)^2 - 4$". How would we go about finding this second form? The fact to notice is that in the second form, x is involved only in an expression which is a perfect square. Therefore, we might ask ourselves the question: How can $x^2 - 2x - 3$ be changed into a form in which x is involved only in a perfect square?

We can do this by "working backward" from "$x^2 - 2x - 3$" as follows:

$$x^2 - 2x - 3 = (x^2 - 2x \quad) - 3 .$$

Now we ask: What is missing inside the parentheses to yield a perfect square? Clearly, what is needed is a "1". Why? Then we have

$$\begin{aligned} x^2 - 2x - 3 &= (x^2 - 2x + 1) - 3 - 1 , \\ &= (x - 1)^2 - 4 . \end{aligned}$$

Why did we also add " -1 " as well as " 1 "?

Let us follow the same procedure with the polynomial " $3x^2 - 12x + 5$"". We have

$$\begin{aligned} 3x^2 - 12x + 5 &= 3(x^2 - 4x \quad) + 5 , \\ &= 3(x^2 - 4x + 4) + 5 - (3)(4) , \\ &= 3(x - 2)^2 - 7 . \end{aligned}$$

Hence, the graph of

$$y = 3x^2 - 12x + 5$$

is a parabola with vertex $(2, -7)$ and axis $x = 2$. It is obtained by moving the graph of " $y = 3x^2$ " to the right 2 units and downward 7 units.

This method of writing a quadratic polynomial in a form in which the variable is involved only in a perfect square is called completing the square. The resulting form is called the standard form of the quadratic polynomial.

Exercises 13 - 2.

1. Put each of the following quadratic polynomials in standard form:

(a) $x^2 - 2x$

(b) $x^2 + x + 1$

(c) $x^2 + 6$

(d) $x^2 - 3x - 2$

(e) $x^2 - 3x + 2$

(f) $5x^2 - 10x - 5$

(g) $4x^2 + 4$

(h) $x^2 + kx$, k a real number

(i) $x^2 + 2x - 1$

(j) $\frac{1}{2}x^2 - 3x + 2$

2. Put each of the following quadratic polynomials in standard form:

(a) $x^2 - x + 2$

(b) $x^2 + 3x + 1$

(c) $3x^2 - 2x$

(d) $(x + 5)(x - 5)$

(e) $6x^2 - x - 15$

(f) $(x + 1 - \sqrt{2})(x + 1 + \sqrt{2})$

3. Describe, without drawing the graphs, the parabolas which are the graphs of the polynomials in problem 2.

4. Draw the graph of

$$y = x^2 + 6x + 5 .$$

In how many points does it cross the x-axis? What points?

5. Draw the graph of

$$y = x^2 + 6x + 9 .$$

In how many points does it cross the x-axis? What points?

6. Draw the graph of

$$y = x^2 + 6x + 13 .$$

In how many points does it cross the x-axis?

7. Solve the equations formed in problems 4 and 5 by letting y have the value 0. Compare the truth sets of these equations with the points where the parabolas cross the x-axis. Devise a rule for determining the points in which a parabola crosses the x-axis.

8. Consider the standard forms of the quadratic polynomials in problems 4, 5 and 6. Which of these polynomials can be factored as the difference of two squares?

13 - 3. Quadratic equations. In problem 7 we learned that the graph of the parabola

$$y = Ax^2 + Bx + C$$

crosses the x-axis at points whose abscissas satisfy the equation

$$Ax^2 + Bx + C = 0.$$

This is called a quadratic equation if $A \neq 0$.

We solved such quadratic equations before in the cases where " $Ax^2 + Bx + C$ " could be factored over the integers. Can " $2x^2 - 3x + 1$ " be factored over the integers? If so, review how you solved the equation

$$2x^2 - 3x + 1 = 0.$$

Now we can go a step farther, because we can write any quadratic polynomial in standard form.

Example 1. Solve the equation

$$x^2 - 2x - 2 = 0 .$$

First, we write the polynomial in standard form:

$$x^2 - 2x - 2 = (x - 1)^2 - 3$$

Is there a real number whose square is 3 ? If so, we may treat the polynomial as the difference of two squares:

$$x^2 - 2x - 2 = (x - 1)^2 - (\sqrt{3})^2$$

Next, recall how to factor the difference of two squares:

$$(x - 1)^2 - (\sqrt{3})^2 = \big((x - 1) + \sqrt{3}\big)\big((x - 1) - \sqrt{3}\big) .$$

Now we have factored the polynomial *over the real numbers*:

$$x^2 - 2x - 2 = \left(x - 1 + \sqrt{3}\right)\left(x - 1 - \sqrt{3}\right)$$

Multiply these factors and verify the product. The final step in the solution is the familiar process of writing the equation " $ab = 0$ " in its equivalent form " $a = 0$ or $b = 0$ " , where a and b are real expressions. Then the sentences

$$x^3 - 2x - 2 = 0 ,$$
$$\left(x - 1 + \sqrt{3}\right)\left(x - 1 - \sqrt{3}\right) = 0 ,$$
$$x - 1 + \sqrt{3} = 0 \quad \text{or} \quad x - 1 - \sqrt{3} = 0 ,$$
$$x = 1 - \sqrt{3} \quad \text{or} \quad x = 1 + \sqrt{3} ,$$

are all equivalent. Hence, the truth set of the equation is $\{1 - \sqrt{3}, \ 1 + \sqrt{3}\}$.

You see that solving a quadratic equation depends on our being able to factor the quadratic polynomial. If the factors are polynomials over the real numbers, we can find the truth set. Furthermore, being able to factor a quadratic polynomial depends upon the standard form of the polynomial. If it is the difference of two squares, we can factor it over the real numbers.

Example 2. Solve the equation

$$x^2 - 2x + 2 = 0 .$$

Writing it in standard form, we have

$$x^2 - 2x + 2 = (x - 1)^2 + 1 .$$

This is not the difference of two squares, and we cannot factor " $x^2 - 2x + 2$ " over the real numbers. The equation

$$(x - 1)^2 + 1 = 0$$

cannot have real number solutions, because $(x - 1)^2$ is always greater than or equal to 0 for every real number x. Why? Hence, $(x - 1)^2 + 1$ is *greater* than 0 for every real number x.

Exercises 13 - 3.

1. Factor the following polynomials over the real numbers, if possible:

   (a) $2(x + 1)^2 - 5$
   (b) $x^2 + 4$
   (c) $6x^2 - x - 15$
   (d) $3(x - 2)^2 + 1$
   (e) $x^2 - 3$
   (f) $9x^2 - 12x + 4$
   (g) $x^2 + 2x - 1$
   (h) $3x - 2x^2$

2. Solve the following quadratic equations:

   (a) $x^2 + 6x + 4 = 0$
   (b) $2x^2 - 5x = 12$
   (c) $x^2 + 4x + 6 = 0$
   (d) $x^2 = 2x + 4$
   (e) $2x^2 = 4x - 11$
   (f) $12x^2 - 8x = 15$

3. Find the coordinates of the vertex of the parabola whose equation is

   $$y = -3x^2 + 6x - 5 .$$

   What is the largest value the polynomial " $-3x^2 + 6x - 5$ " can have?

4. The polynomial " $x^2 - 8x + 21$ " may never have a value less than what positive integer? May it have values greater than this integer? Are all the values of the polynomial integers?

5. Consider the polynomial

$$2x^2 - 4x - 1$$

and its standard form

$$2(x - 1)^2 - 3 .$$

Is 2 the square of a real number? Is 3 ? Hence, " $2(x - 1)^2 - 3$ " is the difference of two squares and can be factored over the real numbers:

$$2(x - 1)^2 - 3 = \left(\sqrt{2}(x - 1) + \sqrt{3}\right)\left(\sqrt{2}(x - 1) - \sqrt{3}\right)$$

What is the truth set of the equation

$$2x^2 - 4x - 1 = 0 ?$$

6. The perimeter of a rectangle is 94 feet, and its area is 496 square feet. What are its dimensions?

7. An open box is constructed from a rectangular sheet of metal 8 inches longer than it is wide as follows: out of each corner a square of side 2 inches is cut, and the sides are folded up. The volume of the resulting box is 256 cubic inches. What were the dimensions of the original sheet of metal?

8. Draw graphs of the following open sentences:

   (a) $y < x^2 + 6x + 5$

   (b) $y = 4$ and $y = 3x^2 - 12x + 13$

   (c) $y > 3x - 2x^2$

   (d) $y = x^2 - 6|x| + 5$.

Chapter 14

Functions

14 - 1. The function concept. Are you good at explaining things to other people? How would you explain to your younger brother exactly how to find the cost of sending a first-class parcel through the mails?

Let us say that you first go to your postmaster and learn these facts about first-class mail: A parcel weighing one ounce or less requires 4 cents postage; if it weighs more than one ounce and less than or equal to two ounces, it requires 8 cents postage, etc. The Post Office will not accept a parcel weighing more than 20 pounds for first-class mailing.

You would probably first explain to your brother that he should weigh his parcel carefully and find the number representing the weight in ounces. To what set of numbers will this number belong? Describe this set exactly. Now you will explain how to determine the amount of postage required. This will be a number in cents. To what set of numbers will this number belong? Describe this set exactly. If your brother's parcel weighs $3\frac{3}{4}$ ounces, what will the postage cost in cents? How much if it weighs 20 pounds and 15 ounces? (Remember the restriction on the weight of the parcel.)

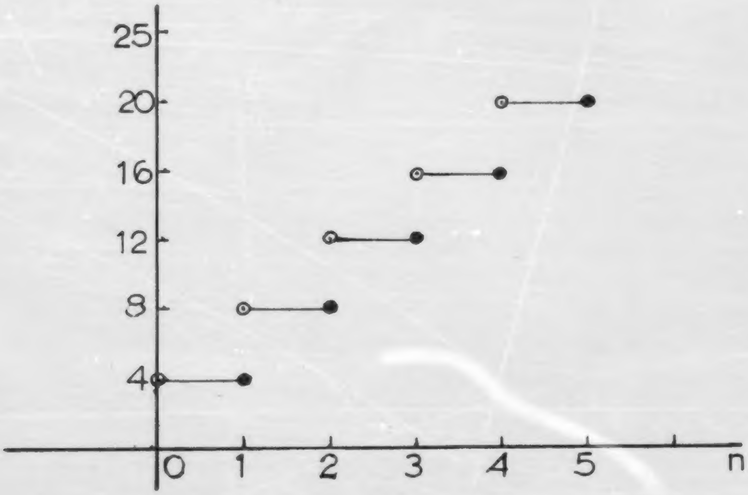
The problem of finding the amount of first-class postage really is a problem of pairing off the numbers of two sets. The numbers of the first set are the real numbers between 0 and 320 representing weights of parcels in ounces. The numbers of the second set are positive integers between 0 and 1280 representing costs of postage in cents. What you are really explaining to your brother is the description of these two sets and the rule which tells him how to take a given number of the first set and associate with it a number of the second set.

Your brother may ask you for a "formula" (to you this would mean an "expression in one variable") which would automatically give him the amount of postage for each number n of ounces. Can you find such a formula which assigns to each real number n between 0 and 320 the number of cents in the required postage? Would

$$4n$$

be such a formula? What is wrong with it?

If you can't find a formula, possibly you can satisfy your brother with a graph which will tell him at a glance what the postage costs. Let us draw a portion of such a graph (for n in ounces from $n = 0$ to $n = 5$):



Interpret the meanings of the circled points and the heavy dots on the graph. How would you explain to your brother how to use this graph to find the number of cents associated with $3\frac{1}{4}$ ounces? With 4 ounces?

Maybe he would understand the postage problem better if you drew up a table for him. Let us say that his scales read to the nearest $\frac{1}{4}$ ounce. You fill in the missing numbers in the table:

First Class Postage

ounces	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{3}{4}$	3	$3\frac{1}{4}$	...
cents														...

You need not feel disappointed in not being able to find a formula for this association. There are many associations of numbers which cannot be described with an expression in one variable. The important point is whether the association can be described in any way, whether it be in terms of a verbal description, a graph, a table, or an expression in one variable.

There are many places in the preceding chapters where we have had occasion in one way or another to associate a real number with each element of a given set. When an idea such as this turns up in such a variety of situations, it becomes worthwhile to separate the idea out and study it carefully for its own sake. It is for this reason that we now make a special study of associations of real numbers of the kind illustrated in the above postage problem. First let us examine some more situations which involve such associations.

Exercises 14 - 1a.

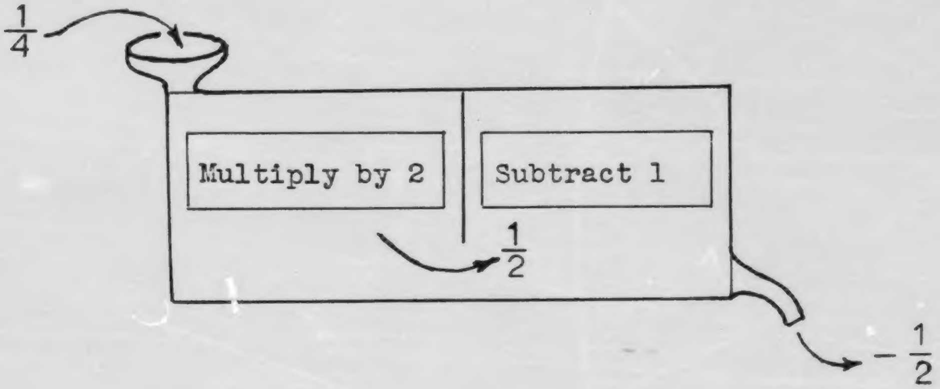
1. In each of the following, describe carefully the two sets and the rule which associates with each element of the first set an element of the second set.

(a)

Positive integer n	1	2	3	4	5	6	7	8	9	10	...
nth odd integer	1	3	5	7	9						

(Fill in the missing numbers. What number is associated with 13 ? With 1000 ?)

(b) Imagine a special computing machine which accepts any positive real number, multiplies it by 2 , subtracts 1 from this, and gives out the result:



(If you feed this machine the number 17, what will come out? What number does the machine associate with 0 ? With -1 ?)

(c) Draw two parallel real number lines and let the unit of measure on the upper line be twice that on the lower line. Then slide the lower line so that its point 1 is directly below the point 1 on the upper line. Now for each point on the upper (first) line there is a point directly below on the second line. (What number is below -13 ? Below 13 ? What number is associated with 1000 by this arrangement?)

(d) Draw a line with respect to a set of coordinate axes such that its slope is 2 and its y-intercept number is -1. For each number a on the x-axis there is a number b on the y-axis such that (a, b) are the coordinates of a point on the given line. (If we pick -1 on the x-axis, the line associates with -1 what number on the y-axis? What number is associated with $-\frac{1}{2}$? With 13 ?)

(e) For each real number t such that $|t| < 1$, use the linear expression "2t - 1" to obtain an associated number. (What number does this expression associate with $-\frac{2}{3}$? With 2 ?)

(f) Given any negative real number, multiply it by 2 and then subtract 1 . (What number does this verbal instruction associate with -13 ? With 0 ?)

2. In each of the following, describe the two sets involved and state verbally the rule which associates the elements of the sets. Tell, in each case, how many elements the rule associates with each element of the first set.

(a) To each real number c such that $c < 1$, assign a number

$$2c - 5 .$$

(b) To each real number d , assign a number e such that (d, e) is a solution of the sentence "$d = |e|$".

(c) To each real number x, assign a number y such that (x, y) is a solution of the equation

$$y = 3x + 7 .$$

(d) To each integer p, assign a number q such that (p, q) is a solution of the sentence "$p > q$".

(e) To each rational number u, assign a number v such that (u, v) is a solution of the equation

$$v^2 = u .$$

3. Give a precise verbal description of the association of weight and cost of a first-class parcel.

4. Describe the workings of a machine that weighs a parcel and automatically places the proper first-class postage on the parcel.

Some associations, such as in problems 2(b), (d), and (e), assign to each number in the first set more than one number in the second set. You will notice, however, that in all our other problems and examples, each number selected from the first set was associated with exactly one number of the second set.

This is the important idea we want to study. We call such an association a function.

> Given a set of numbers and a rule which assigns to each number of this set exactly one number, the resulting association of numbers

is called a function. The given set is called the domain of definition of the function, and the set of assigned numbers is called the range of the function.

We must be extremely careful to understand that two associations are the same function if, and only if, they have the same domain of definition and the same rule. Thus, the functions of problems 1(d) and (e) are the same, even though they are described differently. But the functions of problems 1(a) and (b) are different because they have different domains of definition.

Now we see that a function may be described in many ways: By a table, as in problem 1(a) ; by a machine, as in problem 1(b) ; by a diagram, as in problem 1(c) ; by a graph, as in problem 1(d); by an expression in one variable, as in problem 1(e) ; or by a verbal description, as in problem 1(f) .

For our purpose, the most important way of describing a function is by an expression in one variable, since it allows us to use algebraic methods to study the function. On the other hand, it should be realized that a function need not, and in many cases cannot, be described by an expression in one variable. (Recall the example of the first class postage.) The graphical method is also important because it enables us to visualize certain properties of functions.

Exercises 14 - 1b.

1. Which of the problems in Exercises 13 - 1a describe functions?

If any do not, explain why not.

2. For those problems in Exercises 13 - 1a which describe functions, write if possible the rule in the form of an expression in x, where x belongs to the domain of definition. For example, in problem 1, the rule is given by $5x$, where x is a positive integer.

3. In each of the following, describe (if possible) the function in two ways: (i) by a table, (ii) by an expression in x. In each case, describe the domain of definition.

   (a) With each day associate the income of the ice cream vendor in Chapter 6.

   (b) With each positive integer associate its remainder after division by 5.

   (c) To each positive real number assign $\frac{1}{3}$ of the number increased by 2.

   (d) With each positive integer n associate the n^{th} prime.

   (e) Associate with each day of the year the number of days remaining in the (non-leap) year.

   (f) Associate with the number of dollars invested at 6% for one year the number of dollars earned as interest.

   (g) Associate with each length of the diameter of a circle the length of the circumference.

   (h) Draw two identical parallel number lines and slide the lower line so that its 0 point is directly under the point 1 of the upper line. Then rotate the lower line one-half revolution about its 0 point. Now associate with each number on the upper line the number directly below it on the rotated lower line.

4. With each positive integer associate the smallest factor of the integer (greater than 1). Form a table of ten of the associated pairs given by this function. What integers are associated with themselves?

5. The cost of mailing a package is determined by the weight of the package to the greatest pound. This can be described as: To every positive real number (weight in pounds) assign the integer which is closest to it and greater than or equal to it . Does this describe a function? Can it be represented by an expression in one variable? What is the domain of definition? (Note that the Post Office will not accept a package which weighs more than 32 pounds.) What number does this rule assign to 3.7 ? To 5 ?

6. Assign to each real number x the number -1 if x is rational and the number 1 if x is irrational. What numbers are assigned by this rule to $-\pi$, $-\frac{3}{2}$, $-\frac{\sqrt{2}}{2}$, 0, $\frac{1}{2}$, $\sqrt{2}$, $\frac{\pi}{2}$, 10^6 ? Can you represent this function any way other than by the verbal description?

7. Sometimes the domain of definition of a function is not stated explicitly but is understood to be the largest set of real numbers to which the rule for the function can be

sensibly applied. For example, if a function is described by the expression $\frac{1}{x + 2}$, then, unless stated otherwise, its domain of definition is the set of all real numbers different from -2. (Why?) Similarly the domain of definition of the function defined by $x + 2$ is the set of all real numbers greater than or equal to -2. (Why?) Find the domains of definition of the functions defined by the following expressions:

(a) $\frac{x}{x - 3}$

(b) $\sqrt{2x - 2}$

(c) $3 - \frac{1}{x}$

(d) $\sqrt{x^2}$

(e) $\sqrt{x^2 - 1}$

(f) $\frac{3}{x^2 - 4}$

8. In certain applications, the domain of definition of a function may be automatically restricted to those members which lead to meaningful results in the problem. For example, the area A of a rectangle with fixed perimeter 10 is given by $A = s(5 - s)$, where 5 is the length of a side in feet. The expression $s(5 - s)$ defines a function for all real s, but in this problem we must restrict s to numbers between 0 and 5. (Why?) What are the domains of definition of the functions involved in the following problems?

   (a) What amount of interest is earned by investing x dollars for a year at 4% ?

(b) A triangle has area 12 square inches, and its base measures x inches. What is the length of its altitude?

(c) An open top rectangular box is to be made by cutting a square of side x inches from each corner of a rectangular piece of tin measuring 10" by 8" and then folding up the sides. What is the volume of the box?

14 - 2. The function notation. We have been using letters as names of numbers. In a similar way we shall use letters as names for functions. If f is a given function, and if x is a number in its domain of definition, then we shall designate the number which f assigns to x as $f(x)$. The symbol "$f(x)$" is read "f of x" (it is not f times x), and the number $f(x)$ is called the value of f at x.

The function notation is a very efficient one. Thus, when we wish to describe the function f: "To each real number x assign the real number $2x - 1$", we may write

$$f(x) = 2x - 1, \text{ for each real number } x.$$

Then, $f(\frac{1}{4}) = 2(\frac{1}{4}) - 1 = -\frac{1}{2}$. That is, f assigns to $\frac{1}{4}$ the number $-\frac{1}{2}$.

Similarly $f(0) = 2(0) - 1 = -1$. Also, $f(a) = 2a - 1$ for any real number a.

What real numbers are represented by $f(-\frac{4}{3})$, $f(-\frac{1}{2})$, $f(\frac{4}{3})$, $f(s)$ where s is a real number? If t is a real number, then

$$f(2t) = 2(2t) - 1 = 4t - 1.$$

What real numbers are represented by:

$$f(-t), \quad -f(t), \quad 2f(t), \quad f(t-1), \quad f(t)-1\ ?$$

Sometimes a function is defined in two or more parts, such as the function h defined by

$$h(x) = x, \quad \text{for each number } x \text{ such that } x \geq 0\ ,$$
$$h(x) = -x, \quad \text{for each number } x \text{ such that } x < 0\ .$$

This is a single rule and it defines one function, even though it involves two equations. It is customary to abbreviate this rule to the form

$$h(x) = \begin{cases} x, & x \geq 0\ , \\ -x, & x < 0\ . \end{cases}$$

What is the domain of definition of h? The range of h? Notice that $h(-3) = 3$ and $h(3) = 3$. In fact, we have worked with this function h before in the form

$$h(x) = |x|, \quad \text{for every real number } x.$$

Let us consider another function g defined by the rule:

$$\begin{cases} g(x) = -1, & \text{for each real number } x \text{ such that } x < 0\ , \\ g(x) = 0, & \text{for } x = 0\ , \\ g(x) = 1, & \text{for each real number } x \text{ such that } x > 0\ . \end{cases}$$

It is important to understand that this is also a single rule for a single function which happens to be described in three parts. For convenience in writing, let us again abbreviate the above function g to the form:

$$g(x) = \begin{cases} -1, & x < 0 , \\ 0, & x = 0 , \\ 1, & x > 0 . \end{cases}$$

Notice that g assigns a number to every real number; hence, the domain of definition of g is the set of all real numbers. What is the range of the function? We see that $g(-5) = -1$ and $g(\pi) = 1$. What real numbers are represented by $g(-3.2)$, $g(0)$, $g(\frac{1}{2})$, $g(\sqrt{2})$? If $a > 0$, what is $g(a)$? What is $g(-a)$? If a is any non-zero real number, what is $g(|a|)$? Is it possible to write g in terms of a rule with a single equation, as we did for the function h in the preceding example?

Exercises 14 - 2.

1. Given the function F defined as follows:

$$F(x) = 2 - \frac{x}{2} \quad \text{for each real number } x .$$

What real numbers are represented by:

(a) $F(-2)$
(b) $-F(2)$
(c) $F(-\frac{1}{2}$
(d) $F(1) - 1$
(e) $F(0)$
(f) $|F(-6)|$
(g) $F(|-6|)$
(h) $F(t)$, for any real number t
(i) $F(\frac{t}{2})$
(j) $F(2t)$
(k) $F(\frac{1}{t})$

2. Given the function G defined by:

$$G(t) = |t| \quad \text{for each real number } t.$$

What is the range of G? What real numbers are represented by:

(a) $G(0)$

(b) $G(a) - G(-a)$, for any real number a .

(c) $\frac{G(-3)}{3}$

3. Consider the function h defined by:

$$h(t) = \begin{cases} -1, & t < 0 , \\ 0, & t = 0 , \\ 1, & t > 0 . \end{cases}$$

How does this function h differ from the function g defined in section 14 - 2? (Notice that the same function may have different names and different variables, provided the rule and the domain remain the same.)

4. Consider the function k defined by:

$$k(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 , \\ 0, & x = 0 . \end{cases}$$

Show that k is the same function as the function g defined in section 13 - 2.

5. Given the function H defined by:

$$H(z) = z^2 - 1, \quad -3 < z < 3 .$$

What real numbers are represented by:

(a) $H(2)$ (b) $H(-\frac{1}{3})$

(c) $H(-\frac{1}{3})$

(d) $-H(-2)$

(e) $H(-1) + 1$

(f) $H(3)$

(g) $H(a)$, for any real number a such that $-3 < a < 3$.

(h) $H(t - 1)$, for any real number t such that $-2 < t < 4$.

(i) $H(t) - 1$, for any real number t such that $-3 < t < 3$.

6. Consider the function Q defined by:

$$Q(x) = \begin{cases} -1, & -1 \leq x < 0 , \\ x, & 0 < x \leq 2 . \end{cases}$$

(a) What is the domain of definition of Q ?

(b) What is the range of Q ?

(c) What numbers are represented by $Q(-1)$, $Q(-\frac{1}{2})$, $Q(0)$, $Q(\frac{1}{2})$, $Q(\frac{3}{2})$, $Q(\pi)$?

(d) If R is defined by

$$R(z) = \begin{cases} z, & 0 < z \leq 2 , \\ -1, & -1 \leq z < 0 , \end{cases}$$

is R a different function from Q ?

7. Let F be the function defined in problem 1. What is the truth set of each of the following sentences?

(a) $F(x) = -1$

(b) $F(x) < 0$

(c) $F(x) = -\frac{1}{2}$

(d) $F(x) = x$

(e) $F(x) > 2$

(f) $F(x) \leq 1$

8. Let G be the function defined in problem 2. Draw the graphs of the truth sets of the following sentences:

(a) $G(x) = 1$

(b) $G(x - 1) = 1$

(c) $G(x) \leq 1$

(d) $G(x + 1) > 2$

9. Describe how each of the following pairs of functions differ, if at all:

(a) $f(x) = x - 2$; $F(x) = \frac{x^2 - 4}{x + 2}$

(b) $g(x) = x^2 - 1$; $G(t) = \frac{t^4 - 1}{t^2 + 1}$

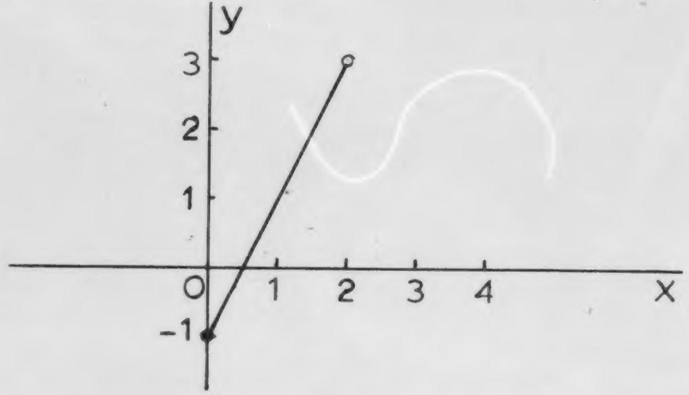
14 - 3. Graphs of functions. One way to represent a function is by means of a graph, as we have seen early in this chapter. When a function f is defined, the graph of f is the graph of the truth set of the equation

$$y = f(x) .$$

Example 1. Draw the graph of the function f defined by:

$$f(x) = 2x - 1, \quad 0 \leq x < 2 .$$

This is the graph of the equation $y = 2x - 1, \ 0 \leq x < 2$.



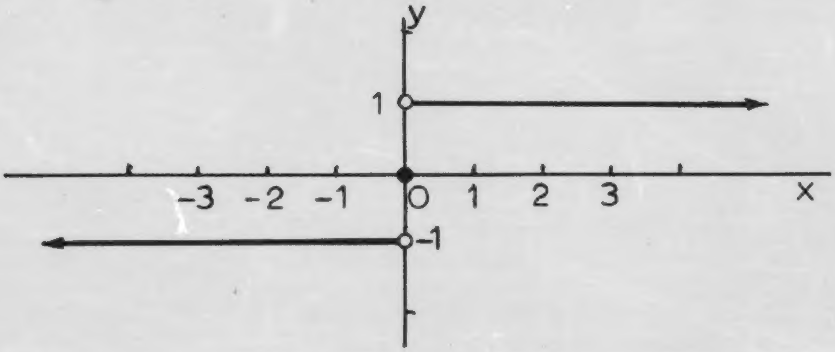
Is this the same as the graph of the function F defined by

$$F(x) = 2x - 1, \quad -2 < x < 2 \ ?$$

Example 2. Draw the graph of the function g defined by:

$$g(x) = \begin{cases} -1, & x < 0 , \\ 0, & x = 0 , \\ 1, & x > 0 . \end{cases}$$

The graph of g is



Exercises 14 - 3a.

1. Draw the graphs of the functions defined as follows:

(a) $T(s) = \frac{1}{3}s + 1, \quad -1 \leq s \leq 2$

(b) $G(x) = |x|, \quad -3 \leq x \leq 3$

(c) $U(x) = \begin{cases} -x, & -3 \leq x < 0 \\ x, & 0 \leq x < 3 \end{cases}$

(d) $V(t) = t^2 - 1, \quad -2 < t \leq 1$

(e) $h(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

(f) $H(z) = \frac{|z|}{z}$

2. What are the domains of definition and the ranges of the functions defined in problem 1 ?

3. Draw the graph of the funcion q defined by

$$q(x) = \begin{cases} -1, & -5 \leq x < -1 , \\ x, & -1 \leq x < 1 , \\ x^2, & 1 < x \leq 2 , \end{cases}$$

4. Give a rule for the definition of the function whose graph is the line extending from (-2, 2) to (4, -1), including endpoints.

5. Give a rule for the definition of the function whose graph consists of two line segments, one extending from (-1, 1) to (0, 0) with end points included, and the other extending from (0, 0) to (2, 1) with end points excluded. What are the domain of definition and range of this function?

6. Draw the graph of a function f which satisfies all of the following conditions over the domain of definition, $-2 \leq x \leq 2$:

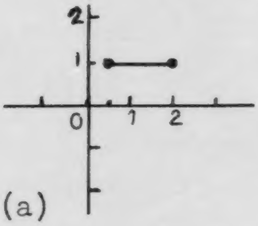
$$f(-1) = 2 ,$$
$$f(0) = 0 ,$$
$$f(1) = 0 ,$$
$$f(2) = 2 ,$$
$$f(x) < 0 \text{ for } 0 < x < 1 .$$

Estimate the value of your function at $\frac{1}{2}$ and at -2 .

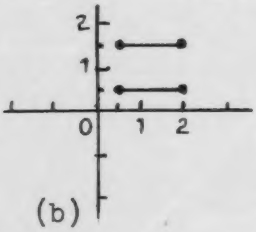
Now that we know how to draw the graph of a function, it is natural to ask whether a given set of points in the plans is the graph of some function. Draw several sets of points and then ask yourself what the definition of a function requires of its graph: It requires that to each abscissa in the domain of definition there be exactly one ordinate assigned by the function. Thus, for each number a in the domain of definition of the function, how many points on its graph have this number as abscissa? If a vertical line is drawn through the graph of a function, in how many points will the line intersect the graph? How would you state the rule for a function in terms of its graph?

Exercises 14 - 3b.

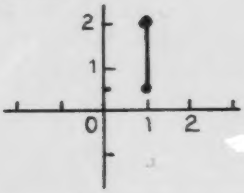
1. Consider the sets of points, coordinate axes not included, indicated in the following figures:



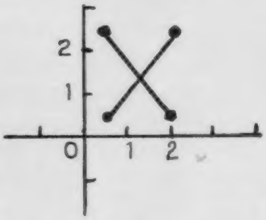
(a)



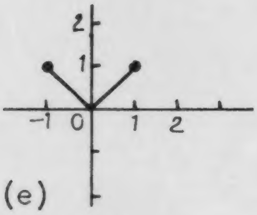
(b)



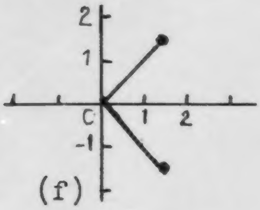
(c)



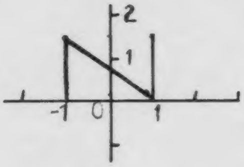
(d)



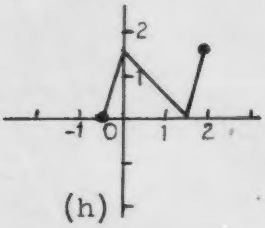
(e)



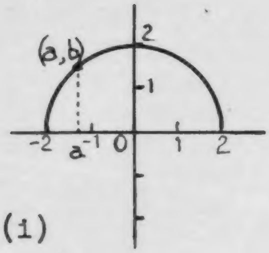
(f)

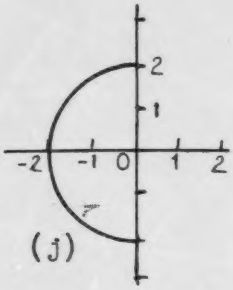


(g)

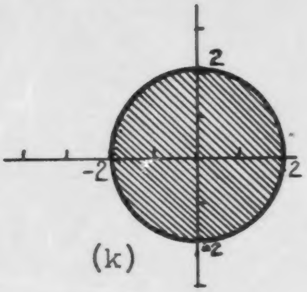


(h)

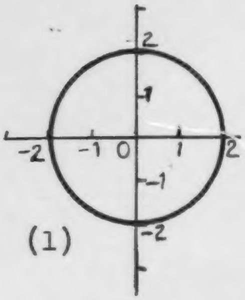

(a,b)
2
1
-2
-1
0
1
2
a
(i)



(j)



(k)



(l)

Which of the above figures is the graph of some function?

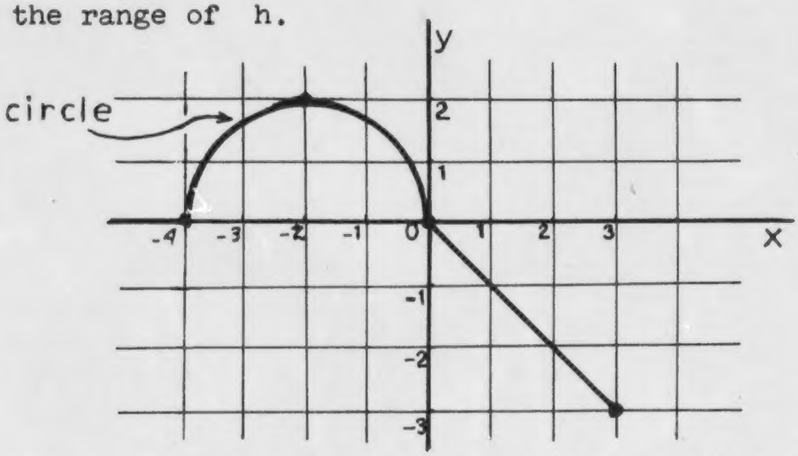
Give the reason for your answer in each case. As an illustration, consider figure (i) . This is the graph of a function f whose domain of definition is the set of all x such that $-2 \leq x \leq 2$. The rule for the function can be stated as follows: If $-2 \leq a \leq 2$, then $f(a) = b$, where (a, b) is the (unique !) point on the graph with abscissa equal to a.

2. The accompanying figure is the graph of a function h. From the graph estimate

   (a) h(-3), y(0), h(2);

   (b) the domain of definition of h;

   (c) the range of h.



3. Let G denote a set of points in the plane which is the graph of some function g.

   (a) For each x in the domain of definition of g, explain how to use the graph to obtain g(x) .

   (b) How do you obtain the domain of definition of g from the graph of G ?

   (c) Show that if (a, b) and (c, d) are any two distinct points of the graph G, then $a \neq c$.

4. Let G be any set of points in the plane with the property that, if (a, b) and (c, d) are any two distinct points of G, then $a \neq c$. Show that G is the graph of a function.
5. Draw the graph of the equation $y^2 = x$, for $0 \leq x < 4$. Is this the graph of some function?

14 - 4. Linear functions. A function whose graph is a straight line (or a portion thereof) is called a linear function. You have already worked with linear functions in Chapter 11, but there they were studied in the form of linear expressions. Can each straight line in the plane be considered as the graph of some linear function? How about the line whose equation is $x = 2$? Can each linear function be represented by an expression in one variable? What is the general form of such an expression? (Recall the y-form of the equation of a line.)

Exercises 14 - 4.

1. If f is a linear function, then there are real numbers A and B such that $f(x) = Ax + B$ for every x in the domain of definition of f.
   (a) Describe the graph of f if $A = 0$.
   (b) Describe the graph of f if $A = 0$ and $B = 0$.
   (c) Determine A and B if the graph of f is the line segment joining (-3, 0) and (1,2), including endpoints.
   (d) What is the domain of definition of the function in part (c) ?

(e) Determine A and B if the graph of f is the line segment joining (-1, 1) and (3, 3) excluding endpoints.

(f) What is the slope and y-intercept of the graph of the function in part (e) ?

(g) What is the domain of definition of the function in part (e) ?

2. If L is the complete line lying on the two points (-3, 1) and (1, -1), describe the function h whose graph consists of the points (x, y) of L such that

$$-2 < y < 2 .$$

3. Let f be the linear function defined by:

$$f(x) = x - 2, \text{ for every real number } x.$$

Which of the following expressions describe a linear function (i.e., with a straight line graph)?

(a) $-(x - 2)$

(b) $|x - 2|$

(c) $\frac{1}{x - 2}$

(d) $|x| - 2$

(e) $(-x) - 2$

(f) $x^2 - 2$

4. Write each expression in problem 3 as a function g in terms of the given function f. Example: The expression (a) describes a function g such that

$$g(x) = -f(x), \text{ for every real number } x.$$

5. How are the graphs of f and g related in problems 3 and 4, parts (a) and (e) ? Draw each pair with reference to a separate set of coordinate axes.

6. If F and G are linear functions defined for every real x by

$$F(x) = -3x + 2, \quad G(x) = 2x - 3,$$

explain how the graph of the sentence

$$\big(y - F(x)\big)\big((y - G(x)\big) = 0$$

is related to the graphs of F and G. (Do this without drawing the graphs of F and G.)

14 - 5. Quadratic functions. We described a linear function in terms of a linear expression in one variable; i.e., any linear function f can be defined by

$$f(x) = Ax + B,$$

where A, B are real numbers. It is natural to define a quadratic function as one which is expressed in terms of a quadratic polynomial in one variable,

$$Ax^2 + Bx + C,$$

where A, B, C are real numbers. If $A = 0$, the quadratic polynomial is reduced to the linear case; hence, we shall assume throughout the remainder of this chapter that $A \neq 0$.

Example 1. Define the function g by:

$$g(x) = 2x^2 - 3x + 1, \text{ for every real number } x.$$

Then

$$g(0) = 1, \quad g(\tfrac{3}{2}) = 2(\tfrac{3}{2})^2 - 3(\tfrac{3}{2}) + 1 = 1,$$

$$g(t) = 2t^2 - 3t + 1, \quad g(2t) = 2(2t)^2 - 3(2t) + 1$$

$$= 8t^2 - 6t + 1,$$

$$g(t - 1) = 2(t - 1)^2 - 3(t - 1) + 1 = 2t^2 - 7t + 6.$$

Also, since $g(x) = (2x - 1)(x - 1)$, it follows that

$$g(-\frac{1}{2}) = 0 \quad \text{and} \quad g(1) = 0. \quad \text{(Why?)}$$

Notice that

$$g(|-2|) = 2|-2|^2 - 3|-2| + 1 = 3,$$

whereas

$$|g(-2)| = |2(-2)^2 - 3(-2) + 1| = 15.$$

Exercises 14 - 5a.

1. Let f be the quadratic function defined by:

$$f(x) = x^2 - 3x - 21, \text{ for every real number } x,$$

and g the quadratic function defined by:

$$g(x) = 3x^2 - 2, \quad -3 < x < 3 .$$

(a) Determine $f(-2)$, $f(-\frac{1}{2})$, $f(0)$, $f(\frac{3}{4})$, $f(3)$;

$f(a)$, $f(\frac{a}{2})$, $f(a + 1)$, where a is any real number.

(b) Determine $g(-2)$, $g(-\frac{1}{2})$, $g(0)$, $g(3)$;

$g(2t - 1)$, $-1 < t < 2$.

(c) Find the truth set of the sentence "$f(x) = 0$."

(d) Draw the graph of the sentence "$f(x) < 0$."

(e) Determine $f(t) + g(t)$, $-3 < t < 3$.

(f) Determine for the real number a,

$f(a) + 3$, $f(a + 3)$, $3f(a)$, $f(3a)$.

(g) Are all the resulting polynomials in part (f) quadratic polynomials in a ?

(h) Determine $f(t)g(t)$, $-3 < t < 3$.

(i) Is the resulting polynomial in (e) a quadratic polynomial in t ? How about the resulting polynomial in (h) ?

2. Describe the functions involved in the following problems. State the domains of definition, and solve the problems.

   (a) What is the area A of a triangle if the length of the base is b inches and the altitude is 10 inches longer than the base?

   (b) What is the product P of two positive numbers if the larger plus twice the smaller, s, is 120 ?

   (c) 120 feet of wire is to be used to build a rectangular pen along the wall of a large barn, the wall of the barn forming one side of the pen. If L is the length of the side of the pen parallel to the wall of the barn, find the area A of the pen.

3. Draw the graph of the quadratic function f defined by:

   (a) $f(x) = x^2 + x - 1$, $-3 \leq x \leq 2$

   (b) $f(x) = 3x^2 - 3$, $-2 < x < 2$

   (c) $f(x) = -x^2 + 1$, $-2 \leq x < 3$

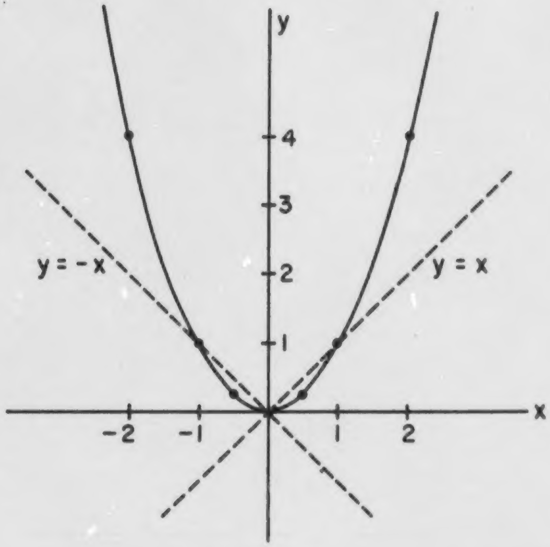


Figure 1.

In problems 4 through 8, refer to Figure 1 and the graph of $y = x^2$, and also refer to Section 11 - 6.

4. For any real number x, $x^2 \leq 0$. (Why?) Also, $x^2 = 0$ if and only if $x = 0$. Explain why the graph of $y = x^2$ lies entirely above the x-axis and touches the x-axis at a single point $(0, 0)$.

5. For any real number x,

$$(-x)^2 = x^2 .$$

If (a, b) is a point on the graph of $y = x^2$, prove that $(-a, b)$ is also on the graphs. (This means that the portion of the graph in Quadrant II can be obtained by rotating the portion in Quadrant I about the y-axis. We say that "the graph of $y = x^2$" is symmetric about the y-axis.)

6. If x is any real number such that $0 < x < 1$, then $x^2 < x$. (Why?) Show that the portion of the graph of $y = x^2$, for $0 < x < 1$, lies below the graph of $y = x$.

7. If $1 < x$, then

$$x < x^2 . \quad \text{(Why?)}$$

Show that the portion of the graph of $y = x^2$, for $1 < x$, lies above the line $y = x$.

8. If a and b are real numbers such that $0 < a < b$, then

$$a^2 < b^2. \quad \text{(Why?)}$$

Show that the graph of $y = x^2$ rises steadily as we move to the right from 0.

9. Show that a horizontal line will intersect the graph of $y = x^2$ in at most two points.

10. Choose any point (a, a^2) on the graph of $y = x^2$. What is the slope of the line from $(0, 0)$ through (a, a^2)? As we choose points of the graph close to the origin (a close to 0) what happens to the slope of this line? Can you explain why the graph of $y = x^2$ is flat near the origin?

Problems 4 through 10 justify the graph of $y = x^2$ drawn in Figure 1. This graph is an example of a parabola. The point $(0, 0)$ is called its vertex, and the line $x = 0$ is called its axis.

With our knowledge of the graph of $y = x^2$, we can obtain graphs of other quadratic functions. This was done in Section 11 - 6 for particular quadratic functions. Let us verify these expensions of the graph of $y = x^2$ to the graphs of $y = Ax^2 + Bx + C$ for real numbers A, B, C, where $A \neq 0$.

Exercises 14 - 5b.

1. Describe how the graph of "$y = ax^2$" differs from the graph of "$y = x^2$" in each of the following cases:

   (a) $0 \leq a < 1$

   (b) $a > 1$

   (c) $-1 < a < 0$

   (d) $a < -1$

   (e) $|a|$ very large.

   (Refer to Figure 2.)

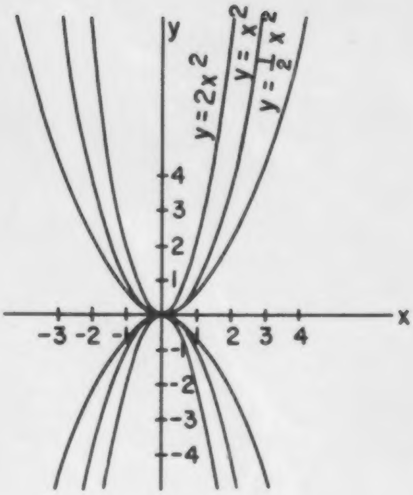


Figure 2.

2. Describe how the graph of $y = x^2 + k$ differs from the graph of $y = x^2$ in each of the following cases:

   (a) $k > 0$

   (b) $k < 0$

   (Refer to Figure 3.)

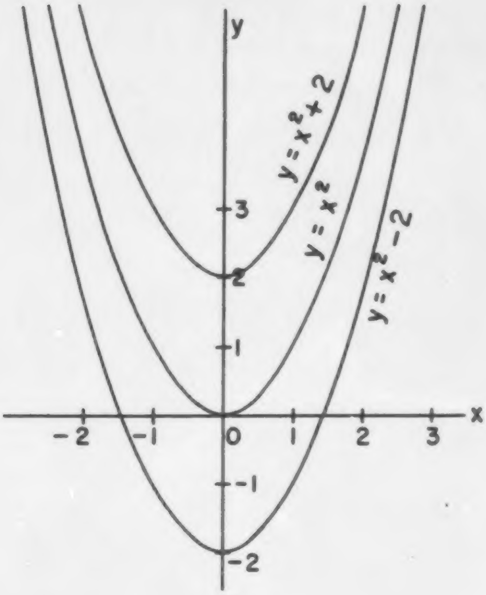


Figure 3.

3. Describe how the graph of $y = (x - h)^2$ differs from the graph of $y = x^2$ in the cases:

   (a) $h > 0$, (b) $h < 0$.

   (Refer to Figure 4.)

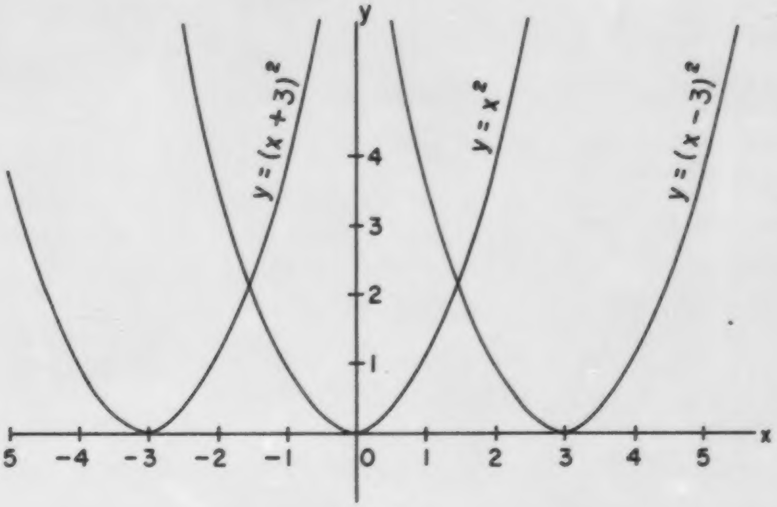


Figure 4.

4. Using the results of problems 1 to 3, describe without drawing their graphs the appearances of the following parabolas:

   (a) $y = (x + 1)^2$ (e) $y = 2(x - 2)^2$

   (b) $y = -3x^2$ (f) $y = (x + 1)^2 + 1$

   (c) $y = x^2 - 3$ (g) $y = 2(x - 1)^2 - 1$

   (d) $y = -(x - 1)^2$ (h) $y = -2(x + 1)^2 - 1$

5. If a, h, k are real numbers, discuss how the graph of $y = a(x - h)^2 + k$ can be obtained from the graph of $y = ax^2$. What is the vertex of the parabola $y = a(x - h)^2 + k$? What is the equation of the axis of this parabola?

6. What is an equation of a parabola whose vertex is $(-1, 1)$ and whose axis is the line $y = 1$? How many parabolas fulfill these conditions?

14 - 6. The graph of $y = Ax^2 + Bx + c$. In Section 11 - 6 we learned how to write a quadratic polynomial $Ax^2 + Bx + C$ in the form $a(x - h)^2 + k$. We called this latter form the standard form of the polynomial. In the preceding section we learned how to draw the graph of the equation $y = a(x - h)^2 + k$. Thus, we have a method for drawing quickly the graph of any quadratic function.

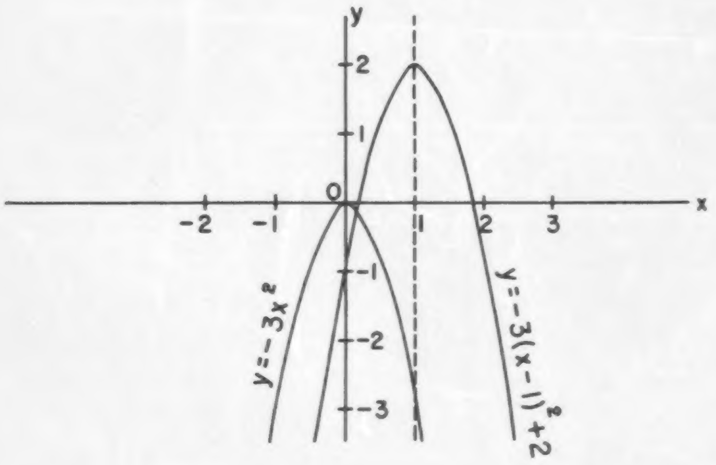
Example 1. Draw the graph of the function f defined by:

$$f(x) = -3x^2 + 6x - 1.$$

By completing the square, we obtain:

$$-3x^2 + 6x - 1 = -3(x^2 - 2x + 1) - 1 + 3 = -3(x - 1)^2 + 2 .$$

The graph of "$y = -3(x - 1)^2 + 2$" is obtained from the graph of "$y = -3x^2$" as indicated:



To verify that the process of moving the graph of "$y = -3x^2$" to the right 1 unit and upward 2 units will actually yield the graph of "$y = -3(x - 1)^2 + 2$, let us argue as follows: Suppose that (a, b) are the coordinates of a point on the graph of the equation

$$y = -3(x - 1)^2 + 2 .$$

Then these coordinates must satisfy the equation; i.e.,

$$b = -3(a - 1)^2 + 2$$

is a true sentence. But then

$$b - 2 = -3(a - 1)^2$$

is also a true sentence. This final sentence asserts that the point with coordinates $(a - 1, b - 2)$ is on the graph of "$y = -3x^2$" . But what are the relative positions of the points $(a - 1, b - 2)$ and (a, b) ? From $(a - 1, b - a)$ we must move 1 unit to the right and 2 units upward to arrive at (a, b) . That is precisely what we did to every point on the graph of "$y = -3x^2$" to arrive at a point on the graph of "$y = -3(x - 1)^2 + 2$" .

By completing squares we changed several quadratic polynomials to the standard form. How do we know that every quadratic polynomial can be put into the standard form? Let us look at the problem in a different way. Given the polynomial $-2x^2 - 4x + 1$, let us find numbers a, h, k (if possible) such that

$$a(x - h)^2 + k = -2x^2 - 4x + 1, \text{ for } \underline{\text{every}} \text{ real number } x.$$

By simplifying the left side, we write

$$ax^2 - 2ahx + (ah^2 + k) = -2x^2 - 4x + 1, \text{ for } \underline{\text{every}} \text{ real number } x.$$

Now we see at a glance that we must find a, h, k so that $a = -2$, $-2ah = -4$, and $(ah^2 + k) = 1$. Why? But if $a = -2$, then the sentence "$-2ah = -4$" is equivalent to "$4h = -4$", i.e., to "$h = -1$". Also, if $a = -2$ and $h = -1$, then "$ah^2 + k = 1$" is equivalent to "$-2 + k = 1$", i.e., to "$k = 3$". With $a = -2$, $h = -1$, and $k = 3$, we have

$$-2(x + 1)^2 + 3 = -2x^2 - 4x + 1, \text{ for every real number } x.$$

This method will work for any quadratic polynomial. Let us try another example.

Example 2. Write the standard form of the polynomial

$$3x^2 - 7x + 5 .$$

We wish to determine real numbers a, h, k such that

$$a(x - h)^2 + k = 3x^2 - 7x + 5, \text{ for every real number } x,$$

i.e., such that

$$ax^2 - 2ahx + (ah^2 + k) = 3x^2 - 7x + 5, \text{ for every real } x.$$

To do this we must find a, h, k such that:

$$a = 3, \quad -2ah = -7, \quad ah^2 + k = 5 .$$

Now, if $a = 3$, then "$-2ah = -7$" is equivalent to "$h = \frac{7}{6}$". Why? If $a = 3$ and $h = \frac{7}{6}$, then "$ah^2 + k = 5$" is equivalent to "$k = \frac{11}{12}$". Why? With $a = 3$, $h = \frac{7}{6}$ and $k = \frac{11}{12}$, we have the standard form:

$$3x^2 - 7x + 5 = 3(x - \tfrac{7}{6})^2 + \tfrac{11}{12}, \text{ for every real } x.$$

Exercises 14 - 6.

1. Write the standard form and draw the graph of each of the following quadratic polynomials:

(a) $x^2 - 6x + 10$

(b) $4x^2 + 4x - 9$

(c) $5x^2 - 3$

(d) $-x^2 - x + \frac{9}{4}$

(e) $4x^2 + 4cx + c^2$

(f) $5x^2 - 3x + \frac{13}{20}$

2. For each quadratic polynomial of problem 1, find the points (if any) where the graph crosses the x-axis.

3. Prove the theorem: Given any quadratic polynomial $Ax^2 + Bx + C$, there exist real numbers a, h, k such that

$$a(x - h)^2 + k = Ax^2 + Bx + C, \text{ for every real number } x .$$

The numbers a, h, k are related to the nu

by the true sentences

$$a = A, \quad h = -\frac{B}{2A}, \quad k = \frac{4AC - B^2}{4A}$$

14 - 7. Solutions of quadratic equations. Consider the three quadratic polynomials

$$x^2 + 2x - 3, \quad x^2 + 2x + 1, \quad x^2 + 2x + 3,$$

and their graphs shown in Figure 5.

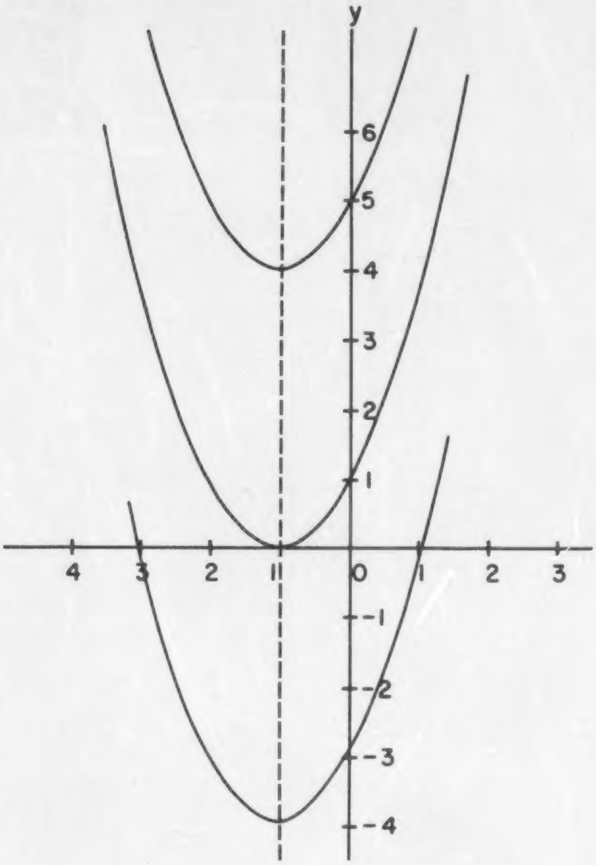


Figure 5.

Notice that the graph of $y = x^2 + 2x - 3$ crosses the x-axis in two points; the graph of $y = x^2 + 2x + 1$ touches the x-axis in a single point; and the graph of $y = x^2 + 2x + 3$ does not intersect the x-axis at all! What is the ordinate of any point on the x-axis? Another way of describing the intersections of these graphs with the x-axis is: The truth set of

$x^2 + 2x - 3 = 0$ is $\{-3, 1\}$,

of $x^2 + 2x + 1 = 0$ is $\{-1\}$,

of $x^2 + 2x + 3 = 0$ is $\emptyset$.

In general, since the graph of

$$y = Ax^2 + Bx + C$$

is always a parabola (if $A \neq 0$), it seems evident that the truth set of the quadratic equation

$$Ax^2 + Bx + C = 0$$

will consist of two, one, or no real numbers according as the parabola intersects, touches, or does not intersect the x-axis.

We have already learned how to solve a quadratic equation whose left side can be factored over the integers. Now consider the quadratic polynomial

$$x^2 + 2x - 1 .$$

We know that it cannot be factored over the integers. (Why?) But it can be written as

$$\begin{aligned} x^2 + 2x - 1 &= (x + 1)^2 - 2 \\ &= (x + 1)^2 - (\sqrt{2})^2 , \end{aligned}$$

i.e., as the difference of two squares. Hence, we may factor $x^2 + 2x - 1$ over the real numbers:

$$x^2 + 2x - 1 = \big((x + 1) + \sqrt{2}\big)\big((x + 1) - \sqrt{2}\big).$$

(Verify this by multiplying the factors.) Then

$$x^2 + 2x - 1 = 0 ,$$

$$(x + 1 + \sqrt{2})(x + 1 - \sqrt{2}) = 0 ,$$

$$x + 1 + \sqrt{2} = 0 \quad \text{or} \quad x + 1 - \sqrt{2} = 0 ,$$

$$x = -1 - \sqrt{2} \quad \text{or} \quad x = -1 + \sqrt{2} ,$$

are all equivalent equations, and the truth set of

$$x^2 + 2x - 1 = 0 \text{ is } \left\{-1 - \sqrt{2},\ -1 + \sqrt{2}\right\}.$$

This example suggests a general procedure for determining whether a quadratic equation has real solutions, and, if so, for finding the solutions. We have shown that any quadratic equation

$$Ax^2 + Bx + C = 0$$

can be written in standard form

$$a(x - h)^2 + k = 0 .$$

Let us assume that a is positive. Otherwise, we may multiply both sides by (-1). The case in which k is a positive number can be disposed of quickly, because we have learned that the graph of $a(x - h)^2 + k$ lies entirely above the x-axis if a and k are positive. Then it cannot cross the x-axis. Hence, there are no real solutions of the equation if $k > 0$.

If $k = 0$, we saw that the graph of $a(x - h)^2$ touches the x-axis at the point $(h, 0)$. Hence, there is one real solution if $k = 0$.

This leaves the case in which k is a negative number. Now we may write

$$a(x - h)^2 + k = a(x - h)^2 - (-k) .$$

If $k < 0$, is there a real number whose square is $(-k)$? How do we factor the difference of two squares? Your result should be

$$a(x - h)^2 - (-k) = (\sqrt{a}(x - h) + \sqrt{-k})(\sqrt{a}(x - h) - \sqrt{-k}) .$$

Thus, if k is negative, the polynomial $a(x - h)^2 + k$ can be factored over the real numbers, and the equation has two real solutions.

Example 1. Factor (a) $2x^2 + 3x - 1$, (b) $x^2 + 3x + 4$.

(a) $$2x^2 + 3x - 1 = 2(x + \frac{3}{2})^2 - \frac{11}{2} \qquad \text{(Verify this.)}$$

$$= 2\ (x + \frac{3}{2})^2 - \frac{11}{4}$$

$$= 2\left((x + \frac{3}{2})^2 - \left(\frac{\sqrt{11}}{2}\right)^2\right)$$

$$= 2\ x + \frac{3}{2} + \frac{\sqrt{11}}{2}\quad x + \frac{3}{2} - \frac{\sqrt{11}}{2} \ .$$

(b) $$x^2 + 3x + 4 = (x + \frac{3}{2})^2 + \frac{7}{4} \ . \qquad \text{(Verify this.)}$$

Here k is a positive number, and we cannot factor this sum of two squares over the real numbers. In fact, $x^2 + 3x + 4$ can never assume the value 0 for any real number x, because $x^2 + 3x + 4$ is the sum of a non-negative number $(x + \frac{3}{2})^2$ and a positive number $\frac{7}{4}$.

Example 2. Solve the equation $x - 3x^2 + 7 = 0$.

The equations $x - 3x^2 + 7 = 0$,

$$3x^2 - x - 7 = 0 \ ,$$

$$3(x - \frac{1}{6})^2 - \frac{85}{12} = 0 \ , \qquad \text{(Why?)}$$

$$(x - \frac{1}{6})^2 - \frac{85}{36} = 0 \ ,$$

$$\left(x - \frac{1}{6} + \frac{\sqrt{85}}{6}\right)\left(x - \frac{1}{6} - \frac{\sqrt{85}}{6}\right) = 0 ,$$

$$x - \frac{1}{6} + \frac{\sqrt{85}}{6} = 0 \quad \text{or} \quad x - \frac{1}{6} - \frac{\sqrt{85}}{6} = 0 ,$$

$$x = \frac{1 - \sqrt{85}}{6} \quad \text{or} \quad x = \frac{1 + \sqrt{85}}{6} ,$$

are all equivalent. Hence, the truth set of

$$x - 3x^2 + 7 = 0 \quad \text{is} \quad \left\{\frac{1 - \sqrt{85}}{6}, \ \frac{1 + \sqrt{85}}{6}\right\}.$$

Exercises 14 - 7.

1. Factor the following quadratic polynomials over the real numbers, if possible:

   (a) $t^2 - 10t + 26$

   (b) $6x^2 - x - 12$

   (c) $\frac{1}{2}x^2 + 4x + 6$

   (d) $4y^2 + 2y + \frac{1}{4}$

   (e) $x^2 + 7x + 14$

   (f) $2 - 2z - z^2$

   (g) $1 - 5x^2$

   (h) $7x^2 - \frac{14}{3}x + \frac{25}{9}$

   (i) $5v^2 - 5v - \frac{11}{4}$

   (j) $x^2 + (a + b)x + ab$, a and b any real numbers.

2. Solve the following quadratic equations:

   (a) $4 - 3x^2 = 0$

   (b) $4 - x - 3x^2 = 0$

   (c) $4 - x + 3x^2 = 0$

   (d) $s^2 - s - \frac{1}{2} = 0$

   (e) $\frac{4}{5}t^2 + \frac{4}{5}t + \frac{1}{5} = 0$

   (f) $\frac{1}{3}y^2 + 2y - 3 = 0$

   (g) $-2y^2 + y - \frac{1}{2} = 0$

   (h) $3n^2 = 7n$

3. Consider the quadratic polynomial in standard form, $a(x - h)^2 + k$, where a, h, k are real numbers and $a = 0$.

   (a) State a rule for deciding whether or not the polynomial can be factored over the real numbers.

   (b) If a, h, k are integers, what conditions on these numbers guarantee that the polynomial can be factored over the integers.

   (c) State a rule for deciding whether the truth set of

   $$a(x - h)^2 + k = 0$$

   contains two, one, or no real numbers.

4. Translate the following into open sentences and solve:

   (a) The perimeter of a rectangle is 12 inches and its area is 7 square inches. What is the length x of its longer side?

   (b) One side of a right triangle is x inches and this side is 1 inch longer than the second side and 2 inches shorter than the hypotenuse. What is the length x?

   (c) The sum of two numbers is 5 and their product is 9. What are the numbers?

5. Consider the general quadratic polynomial $Ax^2 + Bx + C$. Show that

(a) $Ax^2 + Bx + C = A\left(\left(x + \frac{B}{2A}\right)^2 - \frac{B^2 - 4AC}{4A^2}\right)$

(see problem 3 in Exercises 13 - 5c).

(b) If $B^2 - 4AC < 0$, then $Ax^2 + Bx + C = 0$ has no solution.

(c) If $B^2 - 4AC = 0$, then $Ax^2 + Bx + C = 0$ has one solution, $x = -\frac{B}{2A}$.

(d) If $B^2 - 4AC > 0$, then $Ax^2 + Bx + C = 0$ has two solutions,

$$x = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \qquad x = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

(This latter sentence is called the quadratic formula for finding the solutions of the quadratic equation.)